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Expansive Fixed Point Theorems for tri-simulation functions

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Priya Shahi
ORCID: https://orcid.org/0000-0001-7743-4753
St. Andrew's College of Arts, Science and Commerce,
St. Dominic Road, Bandra (W), Mumbai, India,
Email: priya.thaparian@gmail.com
Laxmi Rathour
ORCID: https://orcid.org/0000-0002-2659-7568
Ward Number – 16, Bhagatbandh,
Anuppur 484 224, Madhya Pradesh, India,
Email: laxmirathour817@gmail.com
Vishnu Narayan Mishra
ORCID: https://orcid.org/0000-0002-2159-7710
Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak,
Anuppur 484 4887, Madhya Pradesh, India
Email: vnm@igntu.ac.in

Abstract
In this paper, we introduce the idea of αT-expansion by employing tri-simulation functions introduced by Gubran et al. [R. Gubran, W. M. Alfaqih and M. Imdad, Italian Journal of Pure and Applied Mathematics - N, 45 (2021) 419-430] in a metric space. Further, we shall use these mappings to study various fixed point results in complete metric spaces. The results of this paper generalize and improve several results on the topic in literature.

Keywords: simulation function, α-permisible mapping, fixed point, expansive mapping.

1. Introduction

The classical Banach contraction principle (Banach, 1922), which guarantees the existence and uniqueness of fixed points of contraction self-mappings defined on complete metric spaces while also providing a constructive procedure to compute the fixed point of the underlying mapping, remains an indispensable and effective tool in theory as well as applications within and beyond Mathematics. Many scholars have recently extended this theorem by using more broad contractive mappings on various sorts of spaces. Popa (1997) pioneered the concept of combining many contraction conditions into a single procedure in 1997. To do so, he proposed the implicit function, which is widely used in some works (Ali and Imdad, 2008; Berinde, 2012; Berinde and Vetro, 2012; Imdad et al., 2016; Imdad et al., 2002; and Popa et al., 2010).

Wardowski (2012) is responsible for another notable attempt to expand the Banach contraction principle, in which the author introduced the concept of F-contractions, which has since been investigated in Imdad et al. (2017), Gubranet et al. (2017), and Piri and Kumam, (2016), among other places. Last but not least, Khojasteh et al. (2015) proposed the simulation function, which is also intended to combine many contractions.

It is worth noting that while the concept of simulation function is broad enough to unify numerous existing contractions, it is not relevant to contractions with variables other than $d(Sx, Sy)$ and $d(x, y)$. The only problem that remains is that it treats the equation $\alpha(x, y)d(Sx, Sy)$ as a single element whenever it appears in the contraction inequality.

Recently, Gubran et al. (2021) broaden the scope of simulation functions and addresses the previously described deficiency by allowing the involved terms to occur independently. In particular, they developed a new sort of three-variable simulation function that can be used to unify numerous known contractions from the literature while also being broad enough to create new contractions.

In this paper, we use tri-simulation functions developed by Gubran et al. (2021) in a metric space to present the concept of αT-expansion. Our new idea goes hand in hand with the concept of $\alpha\mathfrak{T}$-contraction introduced by Gubran et al. (2021). These mappings will also be used to investigate various fixed point theorems in entire metric spaces. The findings of this research generalise and improve on various previous studies on the subject.

2. Preliminaries

$\mathbb{R}^+$ stands for the set of non-negative real numbers in the following, while all other terms are used in their standard sense.

Khojasteh et al. (2015) provided the following definitions:

Definition 2.1 Let $\zeta: [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping, then ζ is called a simulation function if it satisfies the following conditions:

$(\zeta 1) \zeta(0,0) = 0;$

$(\zeta 2) \zeta(\mathrm{t}, \mathrm{s}) < \mathrm{s} - \mathrm{t}$ for all $\mathrm{t}, \mathrm{s} > 0;$

$(\zeta 3)$ if $\{\mathrm{t_n}\}, \{\mathrm{s_n}\}$ are sequences in $(0, \infty)$ such that $\lim_{\mathrm{n}\to\infty} \mathrm{t_n} = \lim_{\mathrm{n}\to\infty} \mathrm{s_n} > 0$ then

$$\lim_{\mathrm{n}\to\infty} \sup \zeta(\mathrm{t_n}, \mathrm{s_n}) < 0.$$

We denote the set of all simulation functions by $\mathcal{Z}$.

Definition 2.2 Let (X, d) be a metric space, $\mathrm{T}: \mathrm{X} \to \mathrm{X}$ a mapping and $\zeta \in \mathcal{Z}$. Then T is called a (Z)-contraction with respect to ζ if the following condition is satisfied:

$$\zeta(\mathrm{d}(\mathrm{Tx}, \mathrm{Ty}), \mathrm{d}(\mathrm{x}, \mathrm{y})) \geq 0,$$

for all $\mathrm{x}, \mathrm{y} \in \mathrm{X}$.

Gubran et al. (2021) introduced the following new simulation function involving three variables called as tri-simulation function:

Definition 2.3 Let $\mathrm{T}: \mathbb{R}_+^3 \to \mathbb{R}$ be a mapping. Then T is called a tri-simulation function if it satisfies the following conditions:

(T1): $\mathrm{T}(\mathrm{z}, \mathrm{y}, \mathrm{x}) < \mathrm{x} - \mathrm{yz}$, for all $\mathrm{x}, \mathrm{y} > 0$, $\mathrm{z} \geq 0;$

(T2) : if $\{z_n\}$, $\{y_n\}$ and $\{x_n\}$ are sequences in $(0,\infty)$ such that $y_n < x_n$, for all $n \in \mathbb{N}$, $\lim_{n\to\infty} z_n \geq 1$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n > 0$, then

$$\limsup_{n\to\infty} T(z_n, y_n, x_n) < 0.$$

Gubran et al. (2021) introduced the following new type of contraction:

Definition 2.4 A self-mapping f on a metric space (X, d) is said to be $\alpha\mathfrak{T}$-contraction with respect to $T \in \mathfrak{T}$ if for all $x, y \in X$,

$$T(\alpha(x,y), d(fx, fy), d(x,y)) \geq 0,$$

where $\alpha: M \times M \to \mathbb{R}_+$.

The following two new concepts namely α-permissible and α-orbital permissible mappings were also introduced in Gubran et al. (2021).

Definition 2.5 The mapping f is said to be α-permissible if for all $m \geq n \geq 1$ and $x, y \in X$,

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(f^n x, f^m y) \geq 1.$$

Definition 2.6 The mapping f is said to be α-orbital permissible if for all $m \geq n \geq 1$ and $x \in X$,

$$\alpha(u,\ fu) \geq 1 \Rightarrow \alpha(f^n x,\ f^m v) \geq 1.$$

3. Main Results

We introduce the following definition:

Definition 3.1 A self-mapping f on a metric space (X, d) is said to be αT-expansive with respect to $T \in \mathcal{T}$ if

$$T(\alpha(x,y), d(x,y), \xi(d(fx, fy))) \geq 0, \forall x, y \in X \tag{1}$$

where $\alpha: X \times X \to \mathbb{R}_+$.

Remark 3.1 If f is an αT-expansive for some $T \in \mathfrak{T}$, then by condition (T1), we have

$$\xi(d(fx, fy)) > \alpha(x,y) d(x,y), \tag{2}$$

for all distinct $x, y \in X$.

Our main result in this article runs as follows:

Theorem 3.1 Let (X, d) be a complete metric space and $f: X \to X$ be a bijective, αT-expansive under some tri-simulation function T. Suppose that

(a) f^{-1} is triangular α-permissible;

(b) there exists $x_0 \in X$ such that $\alpha(x_0, f^{-1}x_0) \geq 1$;

(c) f is continuous.

Then f has a fixed point in X.

Proof. Let us define the sequence x_n in X by $x_n = fx_{n+1}$, for all $n \in \mathbb{N}$, where $x_0 \in X$ is such that $\alpha(x_0, f^{-1}x_0) \geq 1$. Since f^{-1} is α-permissible, we have for all $m > n \geq 1$,

$$\alpha(x_n, x_m) \geq 1. \tag{3}$$

If $x_m = x_{m+1}$ for some $m \in \mathbb{N}$, then x_m is a fixed point of f so that we are done. Otherwise, let $d(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$. Then, we have

$$0 \leq T(\alpha(x,y), d(x,y), \xi(d(fx,fy)))$$

$$< \xi(d(fx,fy)) - \alpha(x,y).d(x,y)$$

This implies that

$$\alpha(x,y).d(x,y) < \xi(d(fx,fy)) < d(u_n, u_{n+1}), \tag{4}$$

which shows that $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence of positive real numbers which possesses some limit $r \geq 0$. If $r \neq 0$, then on letting $n \to \infty$ on both sides of the above inequality, we obtain

$$\lim_{n\to\infty} \alpha(x_n, x_{n+1}) = 1$$

In view of (T2), we have

$$0 \leq \lim_{n\to\infty} \sup T(\alpha(x_n, x_{n+1}), d(x_n, x_{n+1}), \xi(d(fx_n, fx_{n+1}))) < 0,$$

a contradiction. Hence, for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{5}$$

Now, we show that $\{x_n\}$ is a bounded sequence. For this, assume that $\{x_n\}$ is unbounded. Then, these exists a subsequence $\{x_{n_k}\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that

$$d(x_{n_k}, x_{n_{k+1}}) > 1 \tag{6}$$

and for $n_k \leq m \leq n_{k+1} - 1$,

$$d(x_{n_k}, x_m) \leq 1.$$

Utilizing the triangular inequality, we have

$$1 < d(x_{n_k}, x_{n_{k+1}})$$

$$\leq d(x_{n_k}, x_{n_{k+1}-1}) + d(x_{n_{k+1}-1}, x_{n_{k+1}})$$

$$\leq 1 + d(x_{n_{k+1}-1}, x_{n_{k+1}}),$$

In view of Remark(3.1), on letting $k \to \infty$ we get

$$\lim_{k\to\infty} d(x_{n_k}, x_{n_{k+1}}) = 1.$$

From the Remark (3.1), we have

$$\alpha(x_{n_k}, x_{n_{k+1}})d(x_{n_k}, x_{n_{k+1}}) < \xi(d(x_{n_k-1}, x_{n_{k+1}-1}))$$

Using (3) and (6), we obtain

$$1 < \alpha(x_{n_k}, x_{n_{k+1}})d(x_{n_k}, x_{n_{k+1}})$$

$$< \xi(d(x_{n_k-1}, x_{n_{k+1}-1}))$$

$$< d(x_{n_k-1}, x_{n_{k+1}-1})$$

$$\leq d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}-1})$$

$$\leq d(x_{n_k-1}, x_{n_k}) + 1$$

which, on letting $k \to \infty$ and in view of Gubran et al. (2021) gives:

$$\lim_{k\to\infty} \alpha(x_{n_k}, x_{n_{k+1}}) = 10.2\text{cmand} \quad 0.2\text{cm}\lim_{k\to\infty} \alpha(x_{n_k-1}, x_{n_{k+1}-1}) = 1$$

Therefore, by (T2), we get

$$0 \leq \limsup_{n\to\infty} T(\alpha(x_{n_k}, x_{n_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), \xi(d(fx_{n_k}, fx_{n_{k+1}}))),$$

which is a contradiction. Therefore, $\{x_n\}$ is a bounded sequence.

Assume that $u_n = \sup\{d(x_i, x_j): i, j \geq n\}$. Observe that, $\{u_n\}$ is decreasing sequence of non-negative real numbers which is bounded due to the boundedness of $\{x_n\}$. So, there exists some $u \geq 0$ such that $\lim_{n\to\infty} u_n = u$. If $u \neq 0$, then by the definition of $\{u_n\}$, for every $k \in \mathbb{N}$ there exist m_k, n_k with $m_k > n_k \geq k$ such that

$$u_k - \frac{1}{k} \leq d(x_{m_k}, x_{n_k}) \leq u_k,$$

which yields

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = c. \tag{7}$$

In view of remark (3.1) and (3), we have

$$d(x_{n_k}, x_{m_k}) \leq \alpha(x_{n_k-1}, x_{m_k-1})d(x_{n_k}, x_{m_k})$$

$$< \xi(d(x_{n_k-1}, x_{m_k-1}))$$

$$< d(x_{n_k-1}, x_{m_k-1})$$

$$\leq d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k-1}),$$

which yields $\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = u$ and $\lim_{k\to\infty} \alpha(x_{n_k}, x_{m_k}) = 1$ as $k \to \infty$. As f is a αT-expansive w.r.t T, we get

$$0 \leq \lim_{n\to\infty} \sup T(\alpha(x_{n_k}, x_{m_k}), d(x_{n_k}, x_{m_k}), \xi(d(x_{n_k-1}, x_{m_k-1}))) < 0,$$

a contradiction which shows that $u = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence. From the completeness of the space X, we conclude that there exists some $v \in X$ such that

$$\lim_{n\to\infty} x_n = v$$

The continuity of the mapping f implies that $\lim_{n\to\infty} x_n = f\lim_{n\to\infty} x_{n-1} = fv$. Hence, $fv = v$.

In what follows, we prove that Theorem 3.1 still holds for T not necessarily continuous, assuming the following condition:

(A): If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\} \to x \in X$ as $n \to +\infty$, then

$$\alpha(f^{-1}x_n, f^{-1}x) \geq 1 \tag{8}$$

for all n.

Theorem 3.2 If in Theorem 3.1 we replace the continuity of f by the condition (A), then the result holds true.

Proof. Following the proof of Theorem 3.1, we know that $\{x_n\}$ is a sequence in X such that $\{x_n\} \to v \in X$ as $n \to +\infty$. From the above condition (A) and Definition 3.1, we get

$$0 \leq T(\alpha(f^{-1}x_{n_k}, f^{-1}v)d(f^{-1}x_{n_k}, f^{-1}v)),$$

so that

$$d(x_{n(k)+1}, f^{-1}v) = d(f^{-1}x_{n(k)}, f^{-1}v)$$

$$\leq \alpha(f^{-1}x_{n(k)}, f^{-1}v)d(f^{-1}x_{n(k)}, f^{-1}v)$$

$$< \xi(d(x_{n(k)}, v)).$$

Letting $k \to \infty$, we obtain $d(v, f^{-1}v) = 0$. This concludes the proof.

Theorem 3.3 *The fixed point of f obtained by Theorem* 3.1 *(or Theorem 3.2) remains unique provided one of the following conditions hold:*

(i) $\alpha(u, v) \geq 1$ for all $u, v \in Fix(f) = \{x \in X: fx = x\}$.

(ii) f is α-permissible and for all $u, v \in X$ there exists $z \in X$ such that $\alpha(u, z) \geq 1$ and $\alpha(v, z) \geq 1$.

Proof. Assume that u and v are two distinct fixed points of f. If the condition (i) is satisfied, then

$$0 \leq T(\alpha(u, v), d(u, v), \xi(d(fu, fv)))$$

$$= T(\alpha(u, v), d(u, v), \xi(d(u, v)))$$

$$< \xi(d(u,v)) - \alpha(u,v)d(u,v)$$

$$< d(u,v) - \alpha(u,v)d(u,v),$$

which is a contradiction so that $u = v$ and hence we are done.

Alternately, if condition (ii) holds, we have $w \in X$ such that $\alpha(u,w) \geq 1$ and $\alpha(v,w) \geq 1$. If one of the two fixed points (say u) is same as w, then we can prove that $w = v$ which leads to contradiction. Thus, we suppose that u, v and w are distinct points. Due to the fact that the function f is α-permissible, we get $\alpha(u,w_n) \geq 1$ and $\alpha(v,w_n) \geq 1$, $\forall n \geq 1$. Now, we need to prove that

$$\lim_{n\to\infty} w_n = u.$$

If $w_m = u$ for some $m \in \mathbb{N}$, then the assertion comes right after that. Otherwise, make the assumption that $d(u,w_n) > 0$ for all $n \in \mathbb{N}$. Now,

$$0 \leq T(\alpha(u_n,w_n), d(u_n,w_n), \xi(d(fu_n,fw_n)))$$

$$< \xi(d(fu_n,fw_n)) - \alpha(u_n,w_n).\, d(u_n,w_n)$$

$$= \xi(d(u,w_{n-1})) - \alpha(u,w_n).\, d(u,w_n)$$

$$< d(u,w_{n-1}) - \alpha(u,w_n).\, d(u,w_n)$$

So, $\{d(u,w_n)\}$ is a strictly decreasing sequence of non-negative real numbers which possesses some limit $r \geq 0$. If $r \neq 0$, then by $(T2)$, we have

$$0 \leq \limsup_{n\to\infty} T(\alpha(u,w_n), d(u,w_n), \xi(d(fu,fw_n))) < 0,$$

a contradiction which substantiates the claim. Similarly, we can show that

$$\lim_{n\to\infty} w_n = v.$$

Now, the uniqueness of the limit implies $u = v$ which brings the proof to a close.

4. Conclusion

In this paper, we established an approach for obtaining expansive fixed point theorems via tri-simulation functions. Many researchers in this field may be prompted to look at new expansive fixed point theorems in metric spaces as a result of this study. These findings can also be applied to more broad spaces like partial metric spaces, b-metric spaces, semi-metric spaces, and other abstract distance spaces.

References

Ali J., Imdad M., (2008), An implicit function implies several contraction conditions, Sarajevo J. Math., **4**, 269–285.

Banach S., (1922), Surles operations dans les ensembles abstraits et leur application aux equations itegrales, Fundamenta Mathematicae, **3,** 133–181.

Berinde V., (2012), Approximating fixed points of implicit almost contractions, Hacet. J. Math. Stat., **41**, 93–102.

Berinde V., Vetro F., (2012), Common fixed points of mappings satisfying implicit contractive conditions, Fixed Point Theory Appl., Art. ID 105.

Gubran R., Alfaqih W.M., Imdad M., (2021), Fixed point results via tri-simulation, Italian Journal of Pure and Applied Mathematics, N, **45,** 419-430.

Gubran R., Imdad M., Khan I.A., Alfaqieh W.M., (2017), Order-theoretic common fixed point results for f-contractions, Bulletin of Mathematical Analysis and Applications, 10, 80–88.

Imdad M., Gubran R., Ahmadullah M., (2016), Using an implicit function to prove common fixed point theorems, arXiv preprint arXiv:1605.05743.

Imdad M., Gubran R., Arif M., Gopal D., (2017), An observation on α-type F-contractions and some ordered-theoretic fixed point results, Mathematical Sciences, **11,** 247–255.

Imdad M., Kumar S., Khan M.S., (2002), Remarks on some fixed point theorems satisfying implicit relations, Rad. Mat, **11,** 135–143.

Khojasteh F., Shukla S., Radenovic′ S., (2015), A new approach to the study of fixed point theory for simulation functions, Filomat **29,** 1189–1194

Piri H., Kumam P., (2016), Wardowski type fixed point theorems in complete metric spaces, Fixed Point Theory Appl., Art. ID 45.

Popa V., (1997), Fixed point theorems for implicit contractive mappings, Stud. Cerc. St. Ser. Mat. Univ. Bacau, **7,** 127–133.

Popa V., Imdad M., Ali J., (2010), Using implicit relations to prove unified fixed point theorems in metric and 2-metric spaces, Bull. Malays. Math. Sci. Soc., **33,** 105–120.

Wardowski D., (2012), Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl., Art. ID 94.