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Computation and Calculus for Combinatorial Geometric Series and Binomial Identities and Expansions

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Abstract
Nowadays, the growing complexity of mathematical and computational modelling demands the simplicity of mathematical and computational equations for solving today's scientific problems and challenges. This paper presents combinatorial geometric series, innovative binomial coefficients, combinatorial equations, binomial expansions, calculus with combinatorial geometric series, and innovative binomial theorems. Combinatorics involves integers, factorials, binomial coefficients, discrete mathematics, and theoretical computer science for finding solutions to the problems in computing and engineering science. The combinatorial geometric series with binomial expansions and its theorems refer to the methodological advances which are useful for researchers who are working in computational science. Computational science is a rapidly growing multi-and inter-disciplinary area where science, engineering, computation, mathematics, and collaboration use advance computing capabilities to understand and solve the most complex real-life problems.

Keywords: computation, combinatorics, binomial coefficient,

1. Introduction

In the earlier days, geometric series served as a vital role in the development of differential and integral calculus and as an introduction to Taylor series and Fourier series. In this article, combinatorial geometric series with binomial expansion and its relationship and theorems are introduced in an innovative way. Combinatorial geometric series is derived from the multiple summations of a geometric series with Annamalai's binomial coefficients. Nowadays, the combinatorial geometric series and its binomial identities and binomial theorems (Annamalai et al., 2022) have significant applications in science, engineering, economics, queuing theory, computation, combinatorics, management, and medicine (Annamalai et al., 2010).

1.1 Geometric Series with Powers of Two

Let us develop the sum of geometric series (Annamalai et al., 2022u, 2022v, 2022w) with exponents of 2 independently as follows:

$$2^n = 2^{n-1} + 2^{n-1} = 2^{n-1} + 2^{n-2} + 2^{n-2} = \cdots = 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots 2^k + 2^k \quad (1)$$

$$\Rightarrow 2^k + 2^{k+2} + 2^{k+3} + \cdots + 2^{n-2} + 2^{n-1} = 2^n - 2^k \Rightarrow \sum_{i=k}^{n} 2^i = 2^{n+1} - 2^k \qquad (2)$$

In the geometric series $if \ \ k = 0$, then $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1, (k, n \ \in N)$,
where $N = \{0, 1, 2, 3, \cdots\}$ is set of natural numbers including zero element.

1.2 Traditional Binomial Coefficient

The factorial function or factorial (Annamalai et al., 2022k, 2022l, 2022w) of a nonnegative integer n, denoted by $n!$, is the product of all positive integers less than or equal to n. For examples, 3! = 1×2× 3 = 6 and 0! = 1.

A binomial coefficient is always an integer that denotes $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$, where $n, r \ \in N$.

Here, $\binom{n+r}{r} = \frac{(n+r)}{r!\,n!} \Rightarrow (n+r) = l \times r!\,n!$, where l is an integer.

2. Binomial Expansions and Combinatorial Geometric Series

When the author of this article was trying to develop the multiple summations of geometric series, a new idea was stimulated his mind for establishing a novel binomial series along with an innovative binomial coefficient (Annamalai et al., 2022r, 2022s, 2022t):

$$\sum_{i_1=0}^{n} \sum_{i_2=i_1}^{n} \sum_{i_3=i_2}^{n} \cdots \sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i \ \& \ V_r^n = \frac{(r+1)(r+2)(r+3)\cdots\cdots(r+n-1)(r+n)}{n!},$$

where $n \geq 1, r \geq 0 \ \ and \ \ n, r \in N$ (3)

Here, $\sum_{i=0}^{n} V_i^r x^i$ and V_r^n refer to the binomial sereis and binomial coefficient respectively.

Let us compare the binomial coefficient V_x^y with the traditional binomial coefficient as follows:

Let $z = x \ + \ y$. Then, $\binom{z}{x} = \ zC_x = \frac{z!}{x!\,y!}$. Here, $V_x^y = V_y^x \Rightarrow \ zC_x = \ zC_y, (x, y, z \in N)$.

For example, $V_3^5 = V_5^3 = \ (5+3)C_3 = \ (5+3)C_5 = 56$.

Also, $V_n^0 = V_0^n = nC_0 = nC_n = \frac{n!}{n!\,0!} = 1 \ \ and \ \ V_0^0 = 0C_0 = \frac{0!}{0!} = 1 (\because 0! = 1)$.

2.1 Computation of Combinatorial Geometric Series

The combinatorial Geometric Series (Annamalai et al., 2022d, 2022e, 2022f) is constituted by double summations of a geometric series as follows:

$$\sum_{i_1=0}^{n} \sum_{i_2=i_1}^{n} x^{i_2} = \sum_{i_2=0}^{n} x^{i_2} + \sum_{i_2=1}^{n} x^{i_2} + \sum_{i_2=2}^{n} x^{i_2} + \cdots + \sum_{i_2=n}^{n} x^{i_2} = 1 + 2x + 3x^2 + \cdots + (n+1)x^n,$$

that is, $$1 + 2x + 3x^2 + \cdots + (n+1)x^n = \sum_{i=0}^{n} (i+1)x^i = \sum_{i=0}^{n} V_i^1 x^i. \qquad (4)$$

The triple summations of a geometric series compute the following combinatorial geometric series:

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n} x^{i_3} = \sum_{i_2=0}^{n}\sum_{i_3=i_2}^{n} x^{i_3} + \sum_{i_2=1}^{n}\sum_{i_3=i_2}^{n} x^{i_3} + \sum_{i_2=2}^{n}\sum_{i_3=i_2}^{n} x^{i_3} + \cdots + \sum_{i_2=n}^{n}\sum_{i_3=i_2}^{n} x^{i_3} = \sum_{i=0}^{n} V_i^2 x^i.$$

Similarly, we can obtain the combinatorial geometric series which is computed by multiple summations of a series.

$$\sum_{i=0}^{n} V_i^r x^i = \sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n} \cdots\cdots\cdots \sum_{i_r=i_{r-1}}^{n} x^{i_r}. \tag{5}$$

Note that the geometirc series $\sum_{i=0}^{n} x^i = \sum_{i=0}^{n} V_i^0 x^i$ is also a combinatorial geometric sereis.

2.2 First Derivative of Geometric Series

Differentiation is the derivative (Annamalai et al., 2022i, 2022j) of a function with respect to an independent variable. In this section, a geometric series is considered as the function of independent variable *x*.

The function of geometric sereis is $f(x) = \sum_{i=0}^{r} x^i = 1 + x + x^2 + x^3 + \cdots + x^r = \dfrac{x^{r+1}-1}{x-1}$.

The first derivative of geometric series is built as follows:

$$f^1(x) = 1 + 2x + 3x^2 + 4x^3 \cdots + rx^{r-1} = f^1\left(\frac{x^{r+1}-1}{x-1}\right) = \frac{(rx-r-1)x^r+1}{(x-1)^2}$$

$$\Rightarrow V_0^1 + V_1^1 x + V_2^1 x^2 + V_3^1 x^3 \cdots + V_{r-1}^1 x^{r-1} = \frac{(rx-r-1)x^r+1}{(x-1)^2}, (x \neq 1).$$

By substituting $x = 2$ in $f^1(x)$, we get the mathematical equation as follows:

$$1 + 2(2) + 3(2)^2 + 4(2)^3 + \cdots + r2^{r-1} = \frac{(r-1)2^r+1}{(2-1)^2} = (r-1)2^r + 1.$$

Similarly, we get the following equations by substituting the values of x:

For $x = 3$, $\quad 1 + 2(3) + 3(3)^2 + 4(3)^3 + \cdots + r3^{r-1} = \dfrac{(2r-1)3^r+1}{(3-1)^2} = \dfrac{(2r-1)3^r+1}{2^2}$.

For $x = 4$, $\quad 1 + 2(4) + 3(4)^2 + 4(4)^3 \cdots + r4^{r-1} = \dfrac{(3r-1)4^r+1}{(4-1)^2} = \dfrac{(3r-1)4^r+1}{3^2}$.

For any number k that is equal to x, we get the equation $\sum_{i=0}^{r-1} V_i^1 k^i = \dfrac{(kr-r-1)k^r+1}{(k-1)^2}$.

2.3 First Derivative of Geometric Series without Differentiation

Sum of the geometric series with negative exponents is that $\sum_{i=1}^{n} x^{-i} = \dfrac{1-x^{-n}}{x-1}$.

The first derivative of geometric series with negative exponents is computed without using differentiation as follows:

$$\sum_{i=1}^{n} x^{-i} = \frac{1-x^{-n}}{x-1} \Rightarrow (-x^{-1})\sum_{i=1}^{n} x^{-i} = -x^{-2} - x^{-3} - x^{-4} - \cdots - x^{-n-1} = (-x^{-1})\frac{1-x^{-n}}{x-1}$$

$$\Rightarrow (-)\sum_{i=1}^{n} x^{-i-1} = -x^{-2} - x^{-3} - x^{-4} - \cdots - x^{-n-1} = \frac{-x^{-1} + x^{-n-1}}{x-1}.$$

Then, $-\sum_{i=1}^{n} x^{-i-1} - \sum_{i=2}^{n} x^{-i-1} - \sum_{i=3}^{n} x^{-i-1} - \cdots - \sum_{i=n}^{n} x^{-i-1}$

$$= \frac{-x^{-1} + x^{-n-1}}{x-1} + \frac{-x^{-2} + x^{-n-1}}{x-1} + \frac{-x^{-3} + x^{-n-1}}{x-1} + \cdots + \frac{-x^{-n} + x^{-n-1}}{x-1}. \quad (6)$$

By simplifying this expression, we get

$$\sum_{i=1}^{n} (-i)x^{-i-1} = \frac{-\sum_{i=1}^{n} x^{-i} + nx^{-n-1}}{x-1} = \frac{\frac{-(1-x^{-n})}{x-1} + nx^{-n-1}}{x-1} = \frac{x^{-n} - 1 + nx^{-n-1}(x-1)}{(x-1)^2}.$$

Thus, $$\sum_{i=1}^{n} (-i)x^{-i-1} = \frac{\big((n+1)x - n\big)x^{-n-1} - 1}{(x-1)^2}, \qquad (x \neq 1). \quad (7)$$

This result denotes the derivative (Annamalai et al., 2022p, 2022q, 2022r) of geometric series with negative exponents.

Next, the sum of the geometric series with nonnegaive exponents is that $\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x-1}$.

The first derivative of geometric series with nonnegative exponents is computed without using differentiation as follows:

$$\sum_{i=0}^{n-1} x^i + \sum_{i=1}^{n-1} x^i + \sum_{i=2}^{n-1} x^i + \cdots + \sum_{i=n-2}^{n-1} x^i + \sum_{i=n-1}^{n-1} x^i$$

$$= \frac{x^n - 1}{x-1} + \frac{x^n - x}{x-1} + \frac{x^n - x^2}{x-1} + \cdots + \frac{x^n - x^{n-2}}{x-1} + \frac{x^n - x^{n-1}}{x-1}.$$

Here, $\sum_{i=0}^{n-1} x^i + \sum_{i=1}^{n-1} x^i + \sum_{i=2}^{n-1} x^i + \cdots + \sum_{i=n-2}^{n-1} x^i + \sum_{i=n-1}^{n-1} x^i = \sum_{i=0}^{n-1} (i+1)x^i$ and

$$\frac{x^n - 1}{x-1} + \frac{x^n - x}{x-1} + \frac{x^n - x^2}{x-1} + \cdots + \frac{x^n - x^{n-2}}{x-1} + \frac{x^n - x^{n-1}}{x-1} = \frac{nx^n - \sum_{i=0}^{n-1} x^i}{x-1}$$

$$= \frac{nx^n - \left(\frac{x^n - 1}{x-1}\right)}{x-1} = \frac{(nx - n - 1)x^n + 1}{(x-1)^2}.$$

Thus, $$\sum_{i=0}^{n-1} (i+1)x^i = \frac{(nx - n - 1)x^n + 1}{(x-1)^2}, \qquad (x \neq 1).$$

Note that $$\sum_{i=k}^{n-1} (i+1)x^i = \frac{\big((n-k)x - (n-k) - 1\big)x^n + x^k}{(x-1)^2}, \qquad (x \neq 1).$$

These results denote the first derivative (Annamalai et al., 2022c, 2022d, 2022e) of geometric series.

2.4 The nth Derivative of Combinatorial Geometric Series

$y = f(x) = \sum_{i=0}^{r} x^i = \frac{x^{r+1}-1}{x-1}$. The derivatives of y are given below.

$$\frac{1}{1!}\frac{dy}{dx} = \sum_{i=0}^{r-1} V_i^1 x^i \Rightarrow \frac{1}{2!}\frac{d^2y}{dx^2} = \sum_{i=0}^{r-2} V_i^2 x^i \Rightarrow \frac{1}{3!}\frac{d^3y}{dx^3} = \sum_{i=0}^{r-3} V_i^3 x^i \Rightarrow \cdots \frac{1}{n!}\frac{d^ny}{dx^n} = \sum_{i=0}^{r-n} V_i^n x^i. \quad (8)$$

The nth derivative [25] of geometric series is

$$\frac{1}{n!}\frac{d^ny}{dx^n} = \sum_{i=0}^{r-n} V_i^n x^i = \frac{1}{n!} f^n(x) = \frac{1}{n!} f^n\left(\frac{x^{r+1}-1}{x-1}\right). \text{Then,} \sum_{i=0}^{r-1} V_i^1 x^i = \frac{1}{1!} f^1\left(\frac{x^{r+1}-1}{x-1}\right);$$

$$\sum_{i=0}^{r-2} V_i^2 x^i = \frac{1}{2!} f^2\left(\frac{x^{r+1}-1}{x-1}\right); \ \& \sum_{i=0}^{r-3} V_i^3 x^i = \frac{1}{3!} f^3\left(\frac{x^{r+1}-1}{x-1}\right) \quad (9)$$

are first, second, and third derivatives respectively.

2.5 Binomial Expansions equal to Multiple of 2

Let us develop some series of binomial coefficients or binomial expansions (Annamalai et al., 2022o, 2022p, 2022q) which are equal to the multiple of 2 or exponents of 2 or both.

$$(i) \sum_{i=0}^{n} V_i^{n-i} = 2^n. \qquad (ii) \sum_{i=0}^{n} i \times V_i^{n-i} = n2^{n-1}. \qquad (iii) \sum_{i=0}^{n} (i+1) V_i^{n-i} = (n+2)2^{n-1}.$$

$$(iv) \sum_{i=0}^{n} (i-1) V_i^{n-i} = (n-2)2^{n-1}, \quad V_r^n = \prod_{i=1}^{n} \frac{(r+i)}{n!}, (n \geq 1, r \geq 0 \,\&\, n, r \in N).$$

2.6 Relations between Binomial Expansion and Combinatorial Geometric Series

Relation 1: $\sum_{i=0}^{n} (i+1) V_i^{n-i} + \sum_{i=0}^{n} (i-1) V_i^{n-i} = \sum_{i=0}^{n} i \times V_i^{n-i} = n2^{n-1}$.

Proof: Let us simplify the general terms in the two parts of binomial expansions as follows:
$(i+1)V_i^{n-i} + (i-1)V_i^{n-i} = 2iV_i^{n-i}$. This idea can be applied to Relation 1.

$$\sum_{i=0}^{n} (i+1) V_i^{n-i} + \sum_{i=0}^{n} (i-1) V_i^{n-i} = 2 \sum_{i=0}^{n} i V_i^{n-i} = (n+2)2^{n-1} + (n-2)2^{n-1} = 2n2^{n-1}.$$

$$\text{Then,} \ 2 \sum_{i=0}^{n} i V_i^{n-i} = 2n2^{n-1} \Rightarrow \sum_{i=0}^{n} i V_i^{n-i} = n2^{n-1}.$$

Hence, Relation 1 is proved.

Relation 2: $\sum_{i=0}^{n} (i+1) V_i^{n-i} - \sum_{i=0}^{n} (i-1) V_i^{n-i} = \sum_{i=0}^{n} V_i^{n-i} = 2^n$.

Proof: Let us simplify the general terms in the two parts of binomial expansions as follows:
$(i+1)V_i^{n-i} - (i-1)V_i^{n-i} = 2V_i^{n-i}$. This idea can be applied to Relation 2.

$$\sum_{i=0}^{n}(i+1)V_i^{n-i} - \sum_{\mathrm{i=0}}^{\mathrm{n}}(\mathrm{i}-1)\mathrm{V_i^{n-i}} = 2\sum_{i=0}^{n}V_i^{n-i} = (n+2)2^{n-1} - (\mathrm{n}-2)2^{\mathrm{n-1}} = 4\times 2^{\mathrm{n-1}}.$$

$$\text{Then, } 2\sum_{i=0}^{n}V_i^{n-i} = 22^{\mathrm{n}} \Rightarrow \sum_{i=0}^{n}V_i^{n-i} = 2^{\mathrm{n}}. \qquad (10)$$

Hence, Relation 2 is proved.

2.7 Annamalai's Binomial Expansion

Let $n, r \in N = \{0,1,2,3,\cdots\}$. The Annamalai's binomial identity is given below:

$$V_0^r + V_1^r + V_2^r + \cdots + V_n^r = V_n^{r+1} \Leftrightarrow V_n^0 + V_n^1 + V_n^2 + \cdots + V_n^r = V_{n+1}^r, (\because V_n^r = V_r^n).$$

From the binomial identity $V_0^r + V_1^r + V_2^r + \cdots + V_n^r = V_n^{r+1}$, we can derive the following binomial expansions:

$$(i). \sum_{i=0}^{n}V_i^0 = \sum_{i=0}^{n}1 = 1+1+1+1+\cdots+1+1 = \frac{(n+1)}{1!}.$$

$$(ii). \sum_{i=0}^{n}V_i^1 = \sum_{i=0}^{n}\frac{(i+1)}{1!} = 1+2+3+\cdots+n+(n+1) = \frac{(n+1)(n+2)}{2!}.$$

$$(iii). \sum_{i=0}^{n}V_i^2 = \sum_{i=0}^{n}\frac{(i+1)(i+2)}{2!} = 1+3+\cdots+\frac{(n+1)(n+2)}{2!} = \frac{(n+1)(n+2)(n+3)}{3!}.$$

$$(iv). \sum_{i=0}^{n}V_i^3 = \sum_{i=0}^{n}\frac{(i+1)(i+2)(i+3)}{3!} = \frac{(n+1)(n+2)(n+3)(n+4)}{4!}.$$

$$(iv). \sum_{i=0}^{n}V_i^4 = \sum_{i=0}^{n}\frac{(i+1)(i+2)(i+3)(i+4)}{4!} = \frac{(n+1)(n+2)(n+3)(n+4)(n+5)}{5!}.$$

Similarly, we can continue this process up to *r* times. The r^{th} binomial expansion is as follows:

$$(r). \sum_{i=0}^{n}V_i^r = \sum_{i=0}^{n}\frac{(i+1)(i+2)(i+3)\cdots(i+r)}{r!} = \frac{(n+1)(n+2)\cdots(n+r)(n+r+1)}{(r+1)!},$$

From the binomial identity $V_n^0 + V_n^1 + V_n^2 + \cdots + V_n^r = V_{n+1}^r$, we can derive the following binomial expansions.

$$(i). \sum_{i=0}^{r}V_0^i = V_1^r \Rightarrow 1+1+1+1+1+\cdots+1+1 = r+1, (\because V_0^r = 1 \text{ for } r = 0,1,2,\cdots).$$

$$(ii). \sum_{i=0}^{r}V_1^i = V_2^r \Rightarrow 1+\frac{2}{1!}+\frac{2\times 3}{2!}+\cdots+\frac{2\times 3\times 4\times\cdots\times r}{r!} = \frac{3\times 4\times 5\times\cdots\times r\times(r+1)}{r!}.$$

$$(iii). \sum_{i=0}^{r}V_2^i = V_3^r \Rightarrow 1+\frac{3}{1!}+\frac{3\times 4}{2!}+\cdots+\frac{3\times 4\times 5\times\cdots\times r}{r!} = \frac{4\times 5\times 6\times\cdots\times r\times(r+1)}{r!}.$$

Similarly, the binomial expansion for $\sum_{i=0}^{r}V_n^i = V_{n+1}^r$ is given below:

$$1+\frac{(n+1)}{1!}+\frac{(n+1)(n+2)}{2!}+\frac{(n+1)(n+2)(n+3)}{3!}+\cdots+\frac{(n+1)(n+2)\cdots(n+r)}{r!}$$
$$=\frac{(n+2)(n+3)\cdots(n+r)(n+r+1)}{r!}.$$

These expressions are called Annamalai's binomial expansions.

2.8 Annamalai's Binomial Identity and Theorem

A binomial theorem (Annamalai et al., 2022e, 2022f, 2022g) is constituted using the Annamalai's binomial identities (Annamalai et al., 2022i, 2022j, 2022k) is given below:

(i) $V_n^0 = V_0^n = 1$ for $n = 0, 1, 2, 3, 3, \cdots$

(ii) $V_r^m = V_m^r$, $\quad (m, r \geq 1 \;\&\; m, r \in N)$.

(iii) $\sum_{i=0}^{r} V_i^n = V_r^{n+1} \quad (OR) \quad \sum_{i=0}^{r} V_n^i = V_{n+1}^r, \quad (\because V_r^m = V_m^r \;\&\; V_n^0 = V_0^n = 1)$.

Theorem 2.1: $\sum_{i=0}^{r} V_i^0 + \sum_{i=0}^{r} V_i^1 + \sum_{i=0}^{r} V_i^2 + \sum_{i=0}^{r} V_i^3 + \cdots + \sum_{i=0}^{r} V_i^n = V_{r+1}^{n+1} - 1.$

Proof. $\sum_{i=0}^{r} V_i^0 = V_r^1; \;\; \sum_{i=0}^{r} V_i^1 = V_r^2; \; \sum_{i=0}^{r} V_i^2 = V_r^3 \;; \; \cdots; \;\; \sum_{i=0}^{r} V_i^n = V_r^{n+1}.$

By adding these expressions on the both sides, we get

$$\sum_{i=0}^{r} V_i^0 + \sum_{i=0}^{r} V_i^1 + \sum_{i=0}^{r} V_i^2 + \sum_{i=0}^{r} V_i^3 + \cdots + \sum_{i=0}^{r} V_i^n = \sum_{i=1}^{n+1} V_r^i$$

Here, $\sum_{i=1}^{n+1} V_r^i = V_r^0 + \sum_{i=1}^{n+1} V_r^i - V_r^0 = \sum_{i=0}^{n+1} V_r^i - 1 = V_{r+1}^{n+1} - 1, (\because V_r^0 = 1).$

$$\therefore \sum_{i=0}^{r} V_i^0 + \sum_{i=0}^{r} V_i^1 + \sum_{i=0}^{r} V_i^2 + \sum_{i=0}^{r} V_i^3 + \cdots + \sum_{i=0}^{r} V_i^n = V_{r+1}^{n+r} - 1. \tag{11}$$

Hence, theorem is proved.

Note that $\sum_{i=0}^{r} V_0^i + \sum_{i=0}^{r} V_1^i + \sum_{i=0}^{r} V_2^i + \sum_{i=0}^{r} V_3^i + \cdots + \sum_{i=0}^{r} V_n^i = V_{n+1}^{r+1} - 1.$

2.9 Combinatorial Geometric Series and Theorem

This Annamalai's binomial expansion is applied into the following binomial series:

$$\sum_{i=0}^{n} V_i^r x^i = \sum_{i=0}^{n} \prod_{j=1}^{r} \frac{i+j}{r!} x^i. \tag{12}$$

The following theorem is derived from the Annamalai's binomial series.

Theorem 2.2: $\sum_{i=0}^{n} V_i^{r+1} x^i = \sum_{i=0}^{n} V_i^r x^i + \sum_{i=1}^{n} V_{i-1}^r x^i + \sum_{i=2}^{n} V_{i-2}^r x^i + \cdots + \sum_{i=n}^{n} V_{i-n}^r x^i. \quad (13)$

Proof: Let's show that the computation of summations of the binomial series (right-hand side of the theorem) is equal to the binomial series (left- hand side of the theorem).

$$\sum_{i=0}^{n} V_i^r x^i + \sum_{i=1}^{n} V_{i-1}^r x^i + \sum_{i=2}^{n} V_{i-2}^r x^i + \cdots + \sum_{i=n-1}^{n} V_{i-(n-1)}^r x^i + \sum_{i=n}^{n} V_{i-n}^r x^i$$

$$= (V_0^r + V_1^r x + V_2^r x^2 + V_3^r x^3 + \cdots + V_n^r x^n) + (V_0^r x + V_1^r x^2 + V_2^r x^3 + V_3^r x^4 + \cdots + V_{n-1}^r x^n)$$

$$+(V_0^r x^2 + V_1^r x^3 + V_2^r x^4 + V_3^r x^5 + \cdots + V_{n-2}^r x^n) + \cdots + (V_0^r x^{n-1} + V_1^r x^n) + V_0^r x^n$$

$$= V_0^r + (V_0^r + V_1^r)x + (V_0^r + V_1^r + V_2^r)x^2 + \cdots + (V_0^r + V_1^r + V_2^r + V_3^r + \cdots + V_n^r)x^n \tag{14}$$

(Note that $V_0^p + V_1^p + V_2^p + \cdots + V_r^p = V_r^{p+1}$for $r = 0, 1, 2, 3, \cdots$, and $V_0^p = V_0^{p+1} = 1$)

$$= V_0^{r+1} + V_1^{r+1} x + V_2^{r+1} x^2 + V_3^{r+1} x^3 + V_4^{r+1} x^4 + \cdots + V_{n-1}^{r+1} x^{n_1} + V_n^{r+1} x^n = \sum_{i=0}^{n} V_i^{r+1} x^i.$$

Hence, theorem is proved.

3. Binomial Expansion equal to the Sum of Geometric Series

Binomial expansion denotes a series of binomial coefficients. In this section, we focus on the summation of multiple binomial expansions or summation of multiple series of binomial coefficients.

Theorem 3.1: $\sum_{i=0}^{0} \binom{0}{i} + \sum_{i=0}^{1} \binom{1}{i} + \sum_{i=0}^{2} \binom{2}{i} + \sum_{i=0}^{3} \binom{3}{i} + \cdots + \sum_{i=0}^{n} \binom{n}{i} = 2^{n+1} - 1.$

This binomial theorem states that the sum of multiple summations of series of binomial coefficients (Annamalai et al., 2022m, 2022n, 2022p) is equal to the sum of a geometric series with exponents of 2.

Proof. Let us find the value of each binomial expansion in the binomial theorem step by step.

Step 0: $\binom{0}{0} = \frac{0!}{0!} = 1 \Rightarrow \sum_{i=0}^{0} \binom{0}{i} = \binom{0}{0} = 2^0.$

Step 1: $\sum_{i=0}^{1} \binom{1}{i} = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2^1.$

Step 2: $\sum_{i=0}^{2} \binom{2}{i} = \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 1 + 2 + 1 = 4 = 2^2.$

Step 3: $\sum_{i=0}^{3} \binom{3}{i} = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8.$

Similarly, we can continue the expressions up to "*step n*" such that $\sum_{i=0}^{n} \binom{n}{i} = 2^n.$

Now, by adding these expressions on both sides, it appears as follows:

$$\sum_{i=0}^{0} \binom{0}{i} + \sum_{i=0}^{1} \binom{1}{i} + \sum_{i=0}^{2} \binom{2}{i} + \sum_{i=0}^{3} \binom{3}{i} + \cdots + \sum_{i=0}^{n} \binom{n}{i} = \sum_{i=0}^{n} 2^i,$$

where $\sum_{i=0}^{n} 2^i = \frac{2^{n+1}-1}{2-1} = 2^{n+1} - 1$ is the geometric sereis with exponents of two.

$$\therefore \sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \sum_{i=0}^{2}\binom{2}{i} + \sum_{i=0}^{3}\binom{3}{i} + \cdots + \sum_{i=0}^{n}\binom{n}{i} = 2^{n+1} - 1.$$

Hence, theorem is proved.

Some results of Theorem 3.1 are given below:

$$(a) \sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \sum_{i=0}^{2}\binom{2}{i} + \sum_{i=0}^{3}\binom{3}{i} + \cdots + \sum_{i=0}^{p-1}\binom{p-1}{i} = 2^p - 1, \text{where } 1 \le p \in N.$$

$$(b) \sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \sum_{i=0}^{2}\binom{2}{i} + \sum_{i=0}^{3}\binom{3}{i} + \cdots + \sum_{i=0}^{p-1}\binom{q-1}{i} = 2^q - 1, \text{where } 1 \le q \in N.$$

By subtracting (a) from (b), we get

$$\left(\sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \cdots + \sum_{i=0}^{q-1}\binom{q-1}{i}\right) - \left(\sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \cdots + \sum_{i=0}^{p-1}\binom{p-1}{i}\right) = 2^q - 2^p,$$

$$i.e., \sum_{i=0}^{p}\binom{p}{i} + \sum_{i=0}^{p+1}\binom{p+1}{i} + \sum_{i=0}^{p+2}\binom{p+2}{i} + \cdots + \sum_{i=0}^{q-2}\binom{q-2}{i} + \sum_{i=0}^{q-1}\binom{q-1}{i} = 2^q - 2^p,$$

where p $< q$ & $p, q \in N$.

By adding (a) and (b), we get

$$\left(\sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \cdots + \sum_{i=0}^{p-1}\binom{p-1}{i}\right) + \left(\sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \cdots + \sum_{i=0}^{q-1}\binom{q-1}{i}\right) = 2^p + 2^q - 2,$$

$$If\ p = q, \qquad then\ 2\left(\sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \cdots + \sum_{i=0}^{p-1}\binom{p-1}{i}\right) = 22^p - 2 = 2(2^p - 1),$$

$$i.e., \quad \sum_{i=0}^{0}\binom{0}{i} + \sum_{i=0}^{1}\binom{1}{i} + \sum_{i=0}^{2}\binom{2}{i} + \sum_{i=0}^{3}\binom{3}{i} + \cdots + \sum_{i=0}^{p-1}\binom{p-1}{i} = 2^p - 1, \text{where } 1 \le q \in N. \quad (15)$$

Theorem 3.2: $\sum_{i=0}^{k}\binom{k}{i} + \sum_{i=0}^{k+1}\binom{k+1}{i} + \sum_{i=0}^{k+2}\binom{k+2}{i} + \cdots + \sum_{i=0}^{n}\binom{n}{i} = 2^{n+1} - 2^k,$

where $k \le n$ & $k, n \in N$.

Proof. The sum of a geometric series with exponents of 2 is given below:

$$\sum_{i=k}^{n} 2^i = 2^{n+1} - 2^k.$$

$$\text{Then,} \sum_{i=0}^{k}\binom{k}{i} + \sum_{i=0}^{k+1}\binom{k+1}{i} + \sum_{i=0}^{k+2}\binom{k+2}{i} + \cdots + \sum_{i=0}^{n}\binom{n}{i} = \sum_{i=k}^{n} 2^i.$$

$$\therefore \sum_{i=0}^{k}\binom{k}{i} + \sum_{i=0}^{k+1}\binom{k+1}{i} + \sum_{i=0}^{k+2}\binom{k+2}{i} + \cdots + \sum_{i=0}^{n}\binom{n}{i} = 2^{n+1} - 2^k. \qquad (16)$$

Hence, theorem is proved.

Some results of Theorem 3.2 are given below:

(i) $\sum_{i=0}^{n}\binom{n}{i} = 2^{n+1} - 2^{n} = 2^{n}$. $\quad (ii) \sum_{i=0}^{n-1}\binom{n-1}{i} + \sum_{i=0}^{n}\binom{n}{i} = 2^{n-1}(2^{2}-1) = 3(2^{n-1})$.

$(iii) \sum_{i=0}^{n-2}\binom{n-2}{i} + \sum_{i=0}^{n-1}\binom{n-1}{i} + \sum_{i=0}^{n}\binom{n}{i} = 2^{n+1} - 2^{n-2} = 2^{n-2}(2^{3}-1) = 7(2^{n-2})$.

$(iv) \sum_{i=0}^{n-3}\binom{n-3}{i} + \sum_{i=0}^{n-2}\binom{n-2}{i} + \sum_{i=0}^{n-1}\binom{n-1}{i} + \sum_{i=0}^{n}\binom{n}{i} = 2^{n+1} - 2^{n-3} = 15(2^{n-3})$.

These results can be generalized as follows:

$$\sum_{i=0}^{p}\binom{p}{i} + \sum_{i=0}^{p+1}\binom{p+1}{i} + \sum_{i=0}^{p+2}\binom{p+2}{i} + \cdots + \sum_{i=0}^{q-1}\binom{q-1}{i} + \sum_{i=0}^{q}\binom{q}{i} = 2^{p}(2^{q-p+1}-1),$$

where $0 \le p \le q \text{ and } p, q \in N$.

Theorem 3.3: $\sum_{i=1}^{1} i\binom{1}{i} + \sum_{i=1}^{2} i\binom{2}{i} + \sum_{i=1}^{3} i\binom{3}{i} + \cdots + \sum_{i=1}^{n} i\binom{n}{i} = (n-1)2^{n} + 1.$

Proof. Let us find the value of each binomial expansion in the binomial theorem step by step.

$Step$ 1: $1\binom{1}{1} = \binom{1}{1} = \frac{1!}{1!\,0!} = 1 \Rightarrow \sum_{i=1}^{1} i\binom{1}{i} = 1 = 1 \times 2^{0}, \qquad (0! = 1).$

$Step$ 2: $\sum_{i=1}^{2} i\binom{2}{i} = 1\binom{2}{1} + 2\binom{2}{2} = 2 + 2 = 4 = 2 \times 2^{1}.$

$Step$ 3: $\sum_{i=1}^{3} i\binom{2}{i} = 1\binom{3}{1} + 2\binom{3}{2} + 3\binom{3}{3} = 3 + 6 + 3 = 12 = 2 \times 2^{1}.$

$Step$ 4: $\sum_{i=1}^{4} i\binom{2}{i} = 1\binom{4}{1} + 2\binom{4}{2} + 3\binom{4}{3} + 4\binom{4}{4} = 4 + 12 + 12 + 4 = 4 \times 2^{3}.$

Similarly, we can continue the expressions up to "*step n*" such that $\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1}$.

Now, by adding these expressions on both sides, it appears as follows:

$$\sum_{i=1}^{1} i\binom{1}{i} + \sum_{i=1}^{2} i\binom{2}{i} + \sum_{i=1}^{3} i\binom{3}{i} + \sum_{i=1}^{4} i\binom{2}{i} + \cdots + \sum_{i=1}^{n} i\binom{n}{i} = \sum_{i=1}^{n} i \times 2^{i-1}.$$

where $\sum_{i=1}^{n} i \times 2^{i} = (n-1)2^{n} + 1$.

$$\therefore \sum_{i=1}^{1} i\binom{1}{i} + \sum_{i=1}^{2} i\binom{2}{i} + \sum_{i=1}^{3} i\binom{3}{i} + \sum_{i=1}^{4} i\binom{2}{i} + \cdots + \sum_{i=1}^{n} i\binom{n}{i} = (n-1)2^{n} + 1. \qquad (17)$$

Hence, theorem is proved.

Some results of Theorem 3.3 are given below:

$$\left\{\sum_{i=1}^{1} i\binom{1}{i}+\sum_{i=1}^{2} i\binom{2}{i}+\sum_{i=1}^{3} i\binom{3}{i}+\cdots+\sum_{i=1}^{k} i\binom{k}{i}+\sum_{i=1}^{k+1} i\binom{k+1}{i}+\cdots+\sum_{i=1}^{n+1} i\binom{n+1}{i}\right\}$$
$$-\left\{\sum_{i=1}^{1} i\binom{1}{i}+\sum_{i=1}^{2} i\binom{2}{i}+\sum_{i=1}^{3} i\binom{3}{i}+\cdots+\sum_{i=1}^{k} i\binom{k}{i}\right\}=n2^{n+1}-k2^{k+1}$$

$$\Rightarrow \sum_{i=1}^{k+1} i\binom{k+1}{i}+\sum_{i=1}^{k+2} i\binom{k+2}{i}+\cdots+\sum_{i=1}^{n} i\binom{n}{i}+\sum_{i=1}^{n+1} i\binom{n+1}{i}=2(n2^{n}-k2^{k}) \text{ and}$$
$$\sum_{i=1}^{k} i\binom{k}{i}+\sum_{i=1}^{k+1} i\binom{k+1}{i}+\cdots+\sum_{i=1}^{n-1} i\binom{n-1}{i}+\sum_{i=1}^{n} i\binom{n}{i}=2\{(n-1)2^{n-1}-(k-1)2^{k-1}\},$$
where $k < n$ & $k, n \in N$.

Theorem 3.4: $(p+1)\int \sum_{i=0}^{n-1} V_i^{p+1} x^i \ dx + C = (p+1)\int \sum_{i=0}^{n-1} V_i^{p+1} x^i \ dx + 1 = \sum_{i=0}^{n} V_i^{p} x^i$,
where C is the constant of Integration and C =1 because 1 is the first term of geometric series.

Proof. Let us prove the theorem on integral calculus using the following binomial expansions.

$$\sum_{i=0}^{n} V_i^{p} x^i = 1+\frac{(p+1)}{1!}x+\frac{(p+1)(p+2)}{2!}x^2+\cdots+\frac{(n+1)(n+2)\ldots(n+p)}{p!}x^n .$$
$$\sum_{i=0}^{n-1} V_i^{p+1} x^i = 1+\frac{(p+2)}{1!}x+\frac{(p+2)(p+3)}{2!}x^2+\cdots+\frac{n(n+1)(n+2)\ldots(n+p)}{(p+1)!}x^{n-1}.$$

Let's prove that the integration (left-hand side of the theorem) is equal to the binomial series (right-hand side of the theorem).

$$\int \sum_{i=0}^{n-1} V_i^{p+1} x^i \ dx = x+\frac{(p+2)}{1!}\frac{x^2}{2}+\frac{(p+2)(p+3)}{2!}\frac{x^3}{3}+\cdots+\frac{n(n+1)\ldots(n+p)}{(p+1)!}\frac{x^n}{n}+C.$$
$$(p+1)\int \sum_{i=0}^{n-1} V_i^{p+1} x^i \ dx = 1+\frac{(p+1)}{1!}x+\frac{(p+1)(p+2)}{2!}x^2+\frac{(p+1)(p+2)(p+3)}{3!}x^3$$
$$+\cdots+\frac{(n+1)(n+2)\ldots(n+p)}{p!}x^n, \qquad \text{where } C=1.$$
$$(p+1)\int \sum_{i=0}^{n-1} V_i^{p+1} x^i \ dx + C = (p+1)\int \sum_{i=0}^{n-1} V_i^{p+1} x^i \ dx + 1 = \sum_{i=0}^{n} V_i^{p} x^i .$$
Hence, theorem is proved.

Some results of Theorem 3.4 are given below:

Let $p = 0$. Then $(p+1)\int \sum_{i=0}^{n-1} V_i^{p+1} x^i \ dx + 1 = \int \sum_{i=0}^{n-1} V_i^{1} x^i \ dx + 1 = \sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}.$

Let $p = 1$. Then $2\int \sum_{i=0}^{n-1} V_i^2 x^i \; dx + 1 = \sum_{i=0}^{n} V_i^1 x^i \Rightarrow \sum_{i=0}^{n-1} V_i^1 x^i = \frac{(rx - r - 1)x^r + 1}{(x-1)^2}$,
which is the first derivative of geometric series. More details about the first derivative of geometric series are given in Section 2.1.

In general., the integration of summation of geometric series is constituted as follows:

$$(p+1)\int \sum_{i=k}^{n-1} V_{i-k}^{p+1} x^i \; dx + C = \sum_{i=k+1}^{n} V_{i-(k+1)}^{p} x^i + V_{i-k}^{p} x^i = \sum_{i=k}^{n} V_{i-k}^{p} x^i \;, \qquad (18)$$

where the integral constant *is* $C = V_{i-k}^{p} x^i$ because it is the first term of the series.

4. Conclusion

In this article, the n^{th} derivative (Annamalai et al., 2022x, 2022y, 2022z) of geometric series has been introduced and its applications used in combinatorics including binomial expansions. Also, computation of the summation of series of binomial expansions and geometric series were derived in an innovative way. Theorems and relations between the binomial expansions and geometric series have been developed for researchers, who are working in science, economics, engineering, and management,

References

Annamalai C (2022) Sum of Summations of Annamalai's Binomial Expansions. *SSRN* 4119994. http://dx.doi.org/10.2139/ssrn.4119994.

Annamalai, C. (2009) A novel computational technique for the geometric progression of powers of two. *Journal of Scientific and Mathematical Research*, Vol.3, pp 16-17. https://doi.org/10.5281/zenodo.6642923.

Annamalai, C. (2010) Application of Exponential Decay and Geometric Series in Effective Medicine. *Advances in Bioscience and Biotechnology*, Vol. 1(1), pp 51-54. https://doi.org/10.4236/abb.2010.11008.

Annamalai, C. (2015) A Theorem on the Annamalai's Binomial Identities. OSF Preprints. https://www.doi.org/10.31219/osf.io/n89xk.

Annamalai, C. (2022a) A Binomial Expansion equal to Multiple of 2 with Non-Negative Exponents. *SSRN* 4116671. http://dx.doi.org/10.2139/ssrn.4116671.

Annamalai, C. (2022b) Algorithmic and Numerical Techniques for Computation of Binomial and Geometric Series. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v8.

Annamalai, C. (2022c) Annamalai's Binomial Identity and Theorem. *SSRN* 4088497. http://dx.doi.org/10.2139/ssrn.4088497.

Annamalai, C. (2022) Calculus and Computation for Geometric Series with Binomial Coefficients. *Cambridge Open Engage*. https://www.doi.org/10.33774/coe-2022-pnx53-v16.

Annamalai, C. (2022d) Combinatorial and Algorithmic Technique for Computation of Binomial Expansions and Geometric Series with its Derivatives. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v10.

Annamalai, C. (2022e) Combinatorial Geometric Series and Binomial Theorems. Cambridge Open Engage. https://www.doi.org/10.33774/coe-2022-pnx53-v17.

Annamalai, C. (2022f) Combinatorial Geometric Series. *OSF Preprints*. https://www.doi.org/10.31219/osf.io/fumzn.

Annamalai, C. (2022g) Computation and Calculus for the Summation of Geometric Series and Binomial Expansions. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v13.

Annamalai, C. (2022h) Computation and combinatorial Techniques for Binomial Coefficients and Geometric Series. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v4.

Annamalai, C. (2022i) Computation and Numerical Method for Summations of Binomial and Geometric Series. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v9.

Annamalai, C. (2022j) Computation Method for Summation of Binomial Expansions equal to Sum of Geometric Series with Exponents of Two. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v6.

Annamalai, C. (2022k) Computation Method for the Summation of Series of Binomial Expansions and Geometric Series with its Derivatives. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v11.

Annamalai, C. (2022l) Computation of Summations of Annamalai's Binomial Expansions. *OSF Preprints.* https://www.doi.org/10.31219/osf.io/cw6hq.

Annamalai, C. (2022m) Computational Method and Calculus for the Summation of Geometric Series and Binomial Expansions. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v15.

Annamalai, C. (2022n) Computational Method for Combinatorial Geometric Series and Binomial Theorems. Cambridge Open Engage. https://www.doi.org/10.33774/coe-2022-pnx53-v20

Annamalai, C. (2022o) Computational Method for Summation of Binomial Expansions equal to Sum of Geometric Series with Exponents of 2. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v5.

Annamalai, C. (2022p) Computational Technique and Differential Calculus for the Summation of Geometric Series and Binomial Expansions. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v12.

Annamalai, C. (2022q) Computational Techniques and Calculus for the Summation of Geometric Series and Binomial Expansions. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v14.

Annamalai, C. (2022r) Computing Method for Binomial Expansions and Geometric Series. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v3.

Annamalai, C. (2022s) Computing Method for Combinatorial Geometric Series and Binomial Expansion. Cambridge Open Engage. https://www.doi.org/10.33774/coe-2022-pnx53-v18.

Annamalai, C. (2022t) Computing Method for Sum of Geometric Series and Binomial Expansions. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v2.

Annamalai, C. (2022u) Differentiation and Integration of Annamalai's Binomial Expansion. *SSRN* 4110255. https://dx.doi.org/10.2139/ssrn.4110255.

Annamalai, C. (2022v) Factorials and Integers for Applications in Computing and Cryptography. *Cambridge Open Engage*. https://www.doi.org/10.33774/coe-2022-b6mks.

Annamalai, C. (2022w) Factorials, Integers and Mathematical and Binomial Techniques for Machine Learning and Cybersecurity. *Cambridge Open Engage*. https://www.doi.org/10.33774/coe-2022-b6mks-v2.

Annamalai, C. (2022w) Novel Binomial Series and its Summations. *SSRN* 4078523. http://dx.doi.org/10.2139/ssrn.4078523.

Annamalai, C. (2022w) Numerical Computational Method for Computation of Binomial Expansions and Geometric Series. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v7.

Annamalai, C. (2022w) Numerical Method and Computation for Combinatorial Geometric Series and Binomial Theorems. *Cambridge Open Engage.* https://www.doi.org/10.33774/coe-2022-pnx53-v19.

Annamalai, C. (2022w) Sum of Geometric Series with Negative Exponents. *SSRN* 4088497. http://dx.doi.org/10.2139/ssrn.4088497.

Annamalai, C. (2022w) Sum of the Summations of Binomial Expansions with Geometric Series. *Cambridge Open Engage*. https://www.doi.org/10.33774/coe-2022-pnx53.

Annamalai, C. (2022x) Summations of Single Terms and Successive Terms of Geometric Series. *SSRN* 408592y2. http://dx.doi.org/10.2139/ssrn.4085922.

Annamalai, C. (2022z) Summations of Single Terms and Successive Terms of Geometric Series. *SSRN* 4085922. http://dx.doi.org/10.2139/ssrn.4085922.