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Nonlinear Thermal Response Analysis of an Internally Heated Radiative-Convective Porous Fin subjected to Magnetic Fields using Method of Lines

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Abstract
This paper demonstrated the computational efficiency and accuracy of method of lines for the nonlinear transient thermal response analysis of a radiative-radiative porous fin with temperature-dependent internal heat generation under the influence of magnetic fields. To establish the computational accuracy of the method, the results of the solution are compared with the results of the developed exact analytical method. Also, the numerical solutions through the method of lines are adopted to explore the impacts of the model parameters on the performance of the passive device. It is found that as the conductive-convective, conductive-radiative, and magnetic field parameters increase, the fin temperature distribution in the fin decreases. The temperature distribution in the fin increases through the fin as the nonlinear thermal conductivity parameter increases. It is hoped that the present study gives a good insight into the nonlinear analysis of the extended surface which will aid the proper design of the extended surfaces in thermal systems.
Keywords: Method of lines. Rectangular porous Fin. Nonlinear analysis. Numerical Investigation. Thermal studies.

Nomenclature

A_{cr} Area of the fin cross section, m^2
B_o magnetic field intensity, Tesla or kg/sec^{2}Amp
c_{pa} specific heat capacity, J/kgK
h coefficient of convective heat transfer, W/m^{2}K
J_c conduction current intensity, A
k fin thermal conductivity, W/mK
k_b fin thermal conductivity at the base temperature, W/mK
L fin length, M
Mc dimensionless convective parameter
Nr dimensionless radiation parameter
P fin perimeter, m
t time, sec.
T fin temperature, K
T_∞ ambient temperature, K
T_b fin temperature at the base, K
x fin axial distance, m
X dimensionless fin length

Greek Symbols

δ fin thickness , m
θ dimensionless temperature
θ_b dimensionless temperature at the fin base
ρ fin material density, kg/m^3
σ Stefan-Boltzmann constant, W/m^{2}K^4
σ Electrical conductivity, Ω^{-1}m^{-1} or sec^{2}Amp^2/kgm^3

1. Introduction

The techno-economically effective cooling of electronics and thermal systems have been achieved through the applications of passive devices such as fins [1-7]. The importance of the extended surfaces has provoked a large volume of research in literatures. The theoretical investigations of thermal damage problems and heat transfer enhancement by the extended surfaces have attest to the facts that the controlling thermal models of the passive devices are always nonlinear. Consequently, the nonlinear thermal models have been successfully analyzed in the past studies with the aids of approximate analytical, semi-analytical, semi-numerical, and numerical methods. In such previous studies, Jordan et al. [8] adopted optimal linearization method to solve the nonlinear problems in the fin while Kundu and Das [9] utilized Frobenius expanding series method for the analysis of the nonlinear thermal model of the fin. Khani et al. [10] and Amirkolaei and Ganji [11] applied homotopy analysis method. In a further analysis, Aziz and Bouaziz [12], Sobamowo [13], Ganji et al. [14] and Sobamowo et al. [15] employed methods of weighted residual to explore the nonlinear thermal behaviour of fins. In another studies, methods of double decomposition and variation of parameter were used by Sobamowo [16] and Sobamowo et al. [17], respectively to study the thermal characteristics of fins. Also, differential transformation method has been used by some researchers such as Moradi and Ahmadikia [18], Sadri et al. [19], Ndlovu and Moitsheki [20], Mosayebidarchech et al. [21], Ghasemi et al. [22] and Ganji and Dogonchi [23] to predict the heat transfer behaviour in the passive devices. With the help of homotopy perturbation method, Sobamowo et al. [24], Arslanturk [25], Ganji et al. [26] and Hoshyar et al. [27] scrutinized the heat flow in the extended surfaces. However, these studies are for thermal analysis of fin under assumed constant heat transfer coefficient. The cases of heat transfer with variable heat transfer coefficient along the passive device varies has also be investigated [28-35]. Such analysis helps in providing the needed information on the efficiency, effectiveness, and design date of the extended surfaces under various boiling modes [33-44].

Although, as pointed out in the review of the previous studies, there are various approximate analytical and numerical solutions that gained applications in solving the thermal problems [45-53]. The adopted numerical methods in the nonlinear problems involve large computational efforts, time and cost while approximate analytical methods involve power series. Indubitably, such power series solutions require rigorous solution procedures with inherent large number of terms which are not convenient for use in practice [15]. Therefore, the quest for simple method with high accurate solutions and prediction cannot be over-emphasized. Moreover, over the years, the simplicity in approach of method of lines and its capability in producing high accurate results to nonlinear transient thermal analysis of extended surfaces have not been demonstrated in the previous studies. Therefore, the present paper focusses on the application of method of lines for numerical investigations of an unsteady nonlinear thermal behaviour of a radiative-convective straight porous fin with temperature-variant thermal conductivity. The improvements and increased accuracy of the prediction as well as the capability of providing good results over a large domain and time by the semi-discretization method is demonstrated in the paper. Also, the computational method shown simplicity in procedures and obviates the inherent complex mathematical analysis, high computational cost, and time in the other methods that have been used to solve the problem. The computational results are verified analytically, and excellent agreements are noted. Also, semi-numerical solutions are used to examine the impacts of the thermal model parameters on performance of fin. The results are presented graphically and discussed.

2. Problem formulation

In Fig. 1, it is consideration is given to a porous fin with temperature-invariant thermal properties allowing radiative and convective heat transfer.

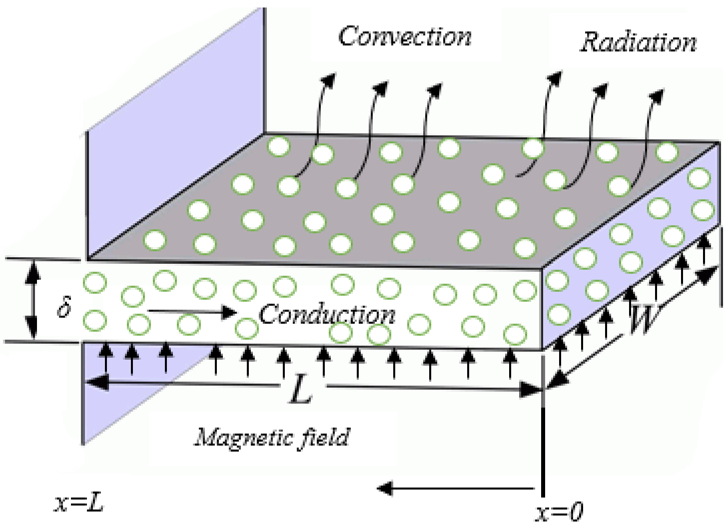

Fig. 1 Schematic of convective-radiative longitudinal fin under even magnetic field

To thermally describe the behaviour of the passive device, assumptions is made that the heat flow porous medium is filled with fluid of single-phase. The solid portion of the extended surface is homogeneous and isotropic. The fin temperature changes only along its length and the condition of a perfect thermal contact between the prime surface and the fin base is assumed.

From the assumptions and with the aid of Darcy's model, the energy balance is

$$\frac{\partial}{\partial \tilde{x}}\left(\frac{d\tilde{T}}{d\tilde{x}}+\frac{4\sigma}{3k_{eff}\beta_R}\frac{\partial \tilde{T}^4}{\partial \tilde{x}}\right)-\frac{\rho\beta c_p gK}{vk_{eff}A_{cr}}(\tilde{T}-T_a)^2-\frac{h(1-\varepsilon)P}{k_{eff}A_{cr}}(\tilde{T}-T_a)$$
$$-\frac{\sigma P\in}{k_{eff}A_{cr}}(\tilde{T}^4-T_a^4)-\frac{\mathbf{J_c}\times\mathbf{J_c}}{\sigma k_{eff}A_{cr}}A_s+\frac{q\left(\tilde{T}\right)}{k_{eff}}=\frac{\rho c_p}{k_{eff}}\frac{\partial \tilde{T}}{\partial \tilde{t}} \tag{1}$$

Expansion of the first term in Eq. (1), it provides

$$\frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2}+\frac{4\sigma}{3k_{eff}\beta_R}\frac{\partial}{\partial \tilde{x}}\left(\frac{\partial \tilde{T}^4}{\partial \tilde{x}}\right)-\frac{\rho\beta c_p gK}{vk_{eff}A_{cr}}(\tilde{T}-T_a)^2-\frac{h(1-\varepsilon)P}{k_{eff}A_{cr}}(\tilde{T}-T_a)$$
$$-\frac{\sigma P\in}{k_{eff}A_{cr}}(\tilde{T}^4-T_a^4)-\frac{\mathbf{J_c}\times\mathbf{J_c}}{\sigma k_{eff}A_{cr}}A_s+\frac{q\left(\tilde{T}\right)}{k_{eff}}=\frac{\rho c_p}{k_{eff}}\frac{\partial \tilde{T}}{\partial \tilde{t}} \tag{2}$$

The initial condition is

when $\tilde{t}=0,\ \tilde{T}=T_0,\quad$ for $\ 0<x<L,$ (3)

The boundary conditions are given as

At $x=0,\ \dfrac{\partial \tilde{T}}{\partial \tilde{x}}=0,\ $ for $\ t>0,$ (4a)

At $x=L,\ \tilde{T}=T_b,\quad$ for $\ t>0,$ (4b)

The internal heat general varies linearly with temperature as

$$q(\tilde{T})=q_a\left(1+\lambda\left(\tilde{T}-T_a\right)\right) \tag{5}$$

When Eq. (5) is substituted into Eq. (2), one arrives at

$$\frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2}+\frac{4\sigma}{3k_{eff}\beta_R}\frac{\partial}{\partial \tilde{x}}\left(\frac{\partial \tilde{T}^4}{\partial \tilde{x}}\right)-\frac{\rho\beta c_p gK}{v\delta k_{eff}}(\tilde{T}-T_a)^2-\frac{h(1-\varepsilon)}{k_{eff}\delta}(\tilde{T}-T_a)$$
$$-\frac{\sigma\in}{k_{eff}\delta}(\tilde{T}^4-T_a^4)-\frac{\mathbf{J_c}\times\mathbf{J_c}}{\sigma k_{eff}A_{cr}}A_s+\frac{q_o}{k_{eff}}\left(1+\lambda\left(\tilde{T}-T_a\right)\right)=\frac{\rho c_p}{k_{eff}}\frac{\partial \tilde{T}}{\partial \tilde{t}} \tag{6}$$

The term T^4 can be expressed as a linear function of temperature as

$$\tilde{T}^4=T_a^4+4T_a^3\left(\tilde{T}-T_a\right)+6T_a^2\left(\tilde{T}-T_\infty\right)^2+...\cong 4T_a^3\tilde{T}-3T_a^4 \tag{7}$$

Substitution of Eq. (7) into Eq. (6), results in

$$\frac{\partial^2\tilde{T}}{\partial\tilde{x}^2}+\frac{16\sigma}{3k_{eff}\beta_R}\frac{\partial^2\tilde{T}}{\partial\tilde{x}^2}-\frac{\rho\beta c_p gK}{v\delta k_{eff}}(\tilde{T}-T_a)^2-\frac{h(1-\varepsilon)}{k_{eff}\delta}(\tilde{T}-T_a)$$
$$-\frac{4\sigma T_a^3\in}{k_{eff}\delta}(\tilde{T}-T_a)-\frac{\mathbf{J_c}\times\mathbf{J_c}}{\sigma k_{eff}A_{cr}}A_s+\frac{q_o}{k_{eff}}\left(1+\lambda\left(\tilde{T}-T_a\right)\right)=\frac{\rho c_p}{k_{eff}}\frac{\partial\tilde{T}}{\partial\tilde{t}} \tag{8}$$

It should be noted that

$$\frac{\mathbf{J_c}\times\mathbf{J_c}}{\sigma}=\sigma_m B_o^2u^2 \tag{9}$$

Therefore

$$\frac{\partial^2\tilde{T}}{\partial\tilde{x}^2}+\frac{16\sigma}{3k_{eff}\beta_R}\frac{\partial^2\tilde{T}}{\partial\tilde{x}^2}-\frac{\rho\beta c_p gK}{v\delta k_{eff}}(\tilde{T}-T_a)^2-\frac{h(1-\varepsilon)}{k_{eff}\delta}(\tilde{T}-T_a)$$
$$-\frac{\sigma T_a^3\in}{k_{eff}\delta}(\tilde{T}-T_a)-\frac{\sigma_m B_o^2u^2}{A_{cr}k_{eff}}A_s+\frac{q_o}{k_{eff}}\left(1+\lambda\left(\tilde{T}-T_a\right)\right)=\frac{\rho c_p}{k_{eff}}\frac{\partial\tilde{T}}{\partial\tilde{t}} \tag{10}$$

Applying the following dimensionless parameters,

$$\tau=\frac{k_b\tilde{t}}{\rho c_p L^2},\; X=\frac{\tilde{x}}{L},\; \theta=\frac{\tilde{T}-T_a}{T_b-T_a}\quad S_h=\frac{\rho\beta c_p gK}{k_{eff}\delta v}L^2\;\; M^2=\frac{h(1-\varepsilon)L^2}{k_{eff}t}\quad Rd=\frac{4\sigma_{st}T_\infty^3}{3\beta_R k_{eff}}\;\; N=\frac{4\sigma_{st}\in L^2T_\infty^3}{k_{eff}t} \tag{11}$$
$$Ha=\frac{\sigma A_s B_o^2u^2L^2}{A_{cr}k_{eff}},\; G=\frac{q_o t}{h\left(T_L-T_\infty\right)}\quad \gamma=\lambda(T_b-T_a)$$

One arrives at the dimensionless form of the governing Eq. (10) as

$$\frac{\partial^2\theta}{\partial X^2}+4Rd\frac{\partial^2\theta}{\partial X^2}-S_h\theta^2-M^2\theta-N\theta-Ha+M^2G\left(1+\gamma\theta\right)=\frac{\partial\theta}{\partial\tau} \tag{12}$$

and the dimensionless initial is given as

When $\tau=0$, $\theta=\theta_0$, for $0<X<1$, (13)

and the dimensionless boundary conditions of the fin are

At $X=0$, $\frac{\partial\theta}{\partial X}=0$, for $\tau>0$, (14a)

At $X=1$, $\theta=1$, for $\tau>0$,
(14b)

Eq. (12) can be written as

$$\frac{\partial^2\theta}{\partial X^2}-\frac{S_h}{1+4Rd}\theta^2-\frac{M^2}{1+4Rd}\theta-\frac{N}{1+4Rd}\theta-\frac{Ha}{1+4Rd}+\frac{M^2G}{1+4Rd}\left(1+\gamma\theta\right)=\frac{\partial\theta}{\partial\tau} \tag{15}$$

Taking

$$Mc^2=\frac{M^2}{1+4Rd}, Nr=\frac{N}{1+4Rd},\ Ra=\frac{S_h}{1+4Rd},\ H=\frac{Ha}{1+4Rd},\ Q=\frac{G}{1+4Rd}, \tag{16}$$

we arrived at the dimensionless forms of the governing as follows;

$$\frac{\partial^2\theta}{\partial X^2}-Ra\theta^2-Mc^2\theta-Nr\theta-H+Mc^2Q+Mc^2Q\gamma\theta=\frac{\partial\theta}{\partial\tau} \tag{17}$$

and the dimensionless boundary conditions still remain the same as in Equation (14).

3. Method of Lines: Its Computational Advantages

Method of lines is a semi-discretization method for solving partial differential equations. It is based on finite difference method to reduce partial differential equation(s) to systems of ordinary differential equations. The method is very simple in the computational approach and possesses superiority of stability advantage over the direct finite difference method [54-67]. In this work, the method is applied to solve the nonlinear equation.

3.1. Application of Method of Lines the Transient Thermal Problems

In order to apply the method of lines, the developed dimensionless differential equation (Eq. 17) is discretized in the space only as follows:

$$\left(\frac{\theta_{i+1}-2\theta_i+\theta_{i-1}}{h^2}\right)-Ra\theta_i^2-(Mc^2+Nr+Mc^2Q\gamma)\theta_i-H+Mc^2Q=\frac{d\theta_i}{d\tau} \tag{18}$$

Which can be written as

$$\frac{d\theta_i}{d\tau}=\left(\frac{\theta_{i+1}-2\theta_i+\theta_{i-1}}{h^2}\right)-Ra\theta_i^2-(Mc^2+Nr+Mc^2Q\gamma)\theta_i-H+Mc^2Q \tag{19}$$

$i=1,2,\ldots,N$

From the initial condition is

$$\theta_i=0,\ \text{ at } \tau=0,\ i=1,2,\ldots,N \tag{20}$$

While for boundary conditions, we have

$$\frac{\theta_2(\tau)-\theta_0(\tau)}{2h}=0\ \Rightarrow\ \theta_2(\tau)=\theta_0(\tau),\quad \rightarrow\theta_0(\tau)=\theta_0\quad at\quad i=0 \tag{21a}$$

and

$$\theta_{N+1}(\tau)=1, \quad \text{at } i=N+1 \tag{21b}$$

where

N is the number of interior node points used in the discretization of the space, x

$h=\dfrac{1}{N+1}$ is the node spacing or space step.

Which can be written as

$$\frac{d\theta_1}{d\tau}=\left(\frac{\theta_2-2\theta_1+\theta_0}{h^2}\right)-Ra\theta_1^2-(Mc^2+Nr+Mc^2Q\gamma)\theta_1-H+Mc^2Q$$

$$\frac{d\theta_2}{d\tau}=\left(\frac{\theta_3-2\theta_2+\theta_1}{h^2}\right)-Ra\theta_2^2-(Mc^2+Nr+Mc^2Q\gamma)\theta_2-H+Mc^2Q$$

$$\frac{d\theta_3}{d\tau}=\left(\frac{\theta_4-2\theta_3+\theta_2}{h^2}\right)-Ra\theta_3^2-(Mc^2+Nr+Mc^2Q\gamma)\theta_3-H+Mc^2Q \tag{22}$$

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$$\frac{d\theta_N}{d\tau}=\left(\frac{\theta_{N+1}-2\theta_N+\theta_{N-1}}{h^2}\right)-Ra\theta_N^2-(Mc^2+Nr+Mc^2Q\gamma)\theta_N-H+Mc^2Q$$

The resulting ODE is generally of the form

$$\frac{d\theta_i}{dt}=A\theta_{i+1}^2+B\theta_{i+1}+C\theta_i+D\theta_{i-1}+E \tag{23}$$

Eq. (23) is a system of N nonlinear first order differential equations and can be written in matrix form as

$$\frac{d\Theta}{dt}=A_M\Phi+A_L\Theta+K \tag{24}$$

where,

A_M and A_L are $N \times N$ matrices given by

$$A_M = Ra\begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix}$$

$$A_L = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix} + (Mc^2 + Nr + Mc^2Q\gamma)\begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix}$$

$$\bar{\Phi} = [\theta_1^2 \quad \theta_2^2 \quad \ldots \quad \theta_N^2]^T$$

$$\Theta = [\theta_1 \quad \theta_2 \quad \ldots \quad \theta_N]^T$$

and

b is a column vector of order $N \times 1$ which is in the form

$$E = \frac{1}{h^2}[\theta_0 \quad 0 \quad \ldots \quad 0 \quad 1]^T + (Mc^2Q - H)[1 \quad 1 \quad \ldots \quad 1 \quad 1]^T$$

Euler’s method is used to solve the system of the nonlinear differential equations. The method is stated as follows

$$\theta_i^{n+1} = \theta_i^n + hf\left(\theta_i^n\right)$$

where

$f\left(\theta_i^n\right) = A\theta_{i+1}^2 + B\theta_{i+1} + C\theta_i + D\theta_{i-1} + E$

h is the time-step

For the **linear thermal model,** the resulting ODE is generally of the form:

$$\frac{d\theta_i}{dt} = B\theta_{i+1} + C\theta_i + D\theta_{i-1} + E \qquad (25)$$

Also, Eq. (25) is a system of N linear first order differential equations and can be written in matrix form as

$$\frac{d\Theta}{dt} = A_L\Theta + b \qquad (26)$$

where,

A_L is a $N \times N$ matrix given by

$$A_L = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}$$

and

$\Theta = [\theta_1 \quad \theta_2 \quad \ldots \quad \theta_N]^T$

b is a column vector of order $N \times 1$ which is in the form

$$b = \frac{1}{h^2}[\theta_0 \quad 0 \quad \ldots \quad 0 \quad \theta_{N+1}]^T = \frac{1}{h^2}[\theta_0 \quad 0 \quad \ldots \quad 0 \quad 1]^T$$

Eigen value and Eigen vector method is used to solve the system of the linear differential equations

4. Development of an Exact Analytical Solution for the Thermal Model

Exact analytical solution for developed ODE (Eq. 17) is also developed for the nonlinear thermal model:

$$\frac{d^2\theta}{dX^2} - Ra\theta^2 - Mc^2\theta - Nr\theta - Ha + Mc^2Q + Mc^2Q\gamma\theta = 0 \tag{27}$$

Using a variable transformation

$$\frac{d\theta}{dX} = \phi, \tag{28}$$

One can write that

$$\frac{d^2\theta}{dX^2} = \frac{d\phi}{dX} = \frac{d\theta}{dX}\frac{d\phi}{d\theta} = \phi\frac{d\phi}{d\theta} \tag{29}$$

Putting Eq. (28) and (29) into Eq. (27), results in

$$\phi\frac{d\phi}{d\theta} - Ra\theta^2 - \left(Mc^2\theta + Nr\theta - Mc^2Q\gamma\right)\theta + Mc^2Q - H = 0 \tag{30}$$

This can easily be written as

$$\phi d\phi + \left\{-Ra\theta^2 - Mc^2\theta - Nr\theta - H + Mc^2Q + Mc^2Q\gamma\theta\right\} d\theta = 0 \tag{31}$$

Eq. (31) is a total differential equation which has a solution of the form

$$\frac{1}{2}\phi^2 + Ra\left(-Mc^2 - Nr - 1\right)\frac{\theta^3}{3} + \left\{Mc^2Q\gamma - \left(Mc^2 - Nr\right)\right\}\frac{\theta^2}{2} + Mc^2Q\theta - H\theta = C \tag{32}$$

where C is the constant of integration

Recall that $\phi = \frac{d\theta}{dX} \quad \rightarrow \phi^2 = \left(\frac{d\theta}{dX}\right)^2$

Therefore, Eq. (32) can be expressed as follows

$$\frac{1}{2}\left(\frac{d\theta}{dX}\right)^2 + Ra\left(-Mc^2 - Nr - 1\right)\frac{\theta^3}{3} + \left\{Mc^2Q\gamma - \left(Mc^2 - Nr\right)\right\}\frac{\theta^2}{2} + Mc^2Q\theta - H\theta = C \quad (33)$$

Using the first boundary condition,

$$X = 1, \quad \frac{d\theta}{dX} = 0 \quad \rightarrow X = 1, \quad \theta = \theta_o,$$

Therefore, the constant C as

$$Ra\left(-Mc^2 - Nr - 1\right)\frac{\theta_0^3}{3} + \left\{Mc^2Q\gamma - \left(Mc^2 + Nr\right)\right\}\frac{\theta_0^2}{2} + \left(Mc^2Q - H\right)\theta_0 = C \quad (34)$$

where θ_o is the dimensionless temperature at the tip.

When Eq. (34) is substituted into Eq. (33), we have

$$\frac{1}{2}\left(\frac{d\theta}{dX}\right)^2 - Ra\left(-Mc^2 - Nr - 1\right)\left(\frac{\theta_0^3}{3} - \frac{\theta^3}{3}\right) + \left\{Mc^2Q\gamma - \left(M^2 + Nr\right)\right\}\left(\frac{\theta_0^2}{4} - \frac{\theta^2}{4}\right) - \left(Mc^2Q - H\right)\left(\theta_0 - \theta\right) = 0$$

(35)

On re-arranging

$$\frac{1}{2}\left(\frac{d\theta}{dX}\right)^2 = Ra\left(-Mc^2 - Nr - 1\right)\left(\frac{\theta_0^3}{3} - \frac{\theta^3}{3}\right) + \left(Mc^2Q - H\right)\left(\tilde{\theta}_0 - \theta\right) + \left\{Mc^2Q\gamma - \left(M^2 + Nr\right)\right\}\left(\frac{\theta_0^2}{4} - \frac{\theta^2}{4}\right)$$

(36)

Eq. (36) is expressed as

$$\sqrt{2}dX = \frac{d\theta}{\sqrt{\left\{Ra\left(Mc^2+Nr+1\right)\left(\frac{\theta^3}{3}-\frac{\theta_0^3}{3}\right)-\left(Mc^2Q-H\right)\left(\theta-\theta_0\right)-\left\{Mc^2Q\gamma-\left(M^2+Nr\right)\right\}\left(\frac{\theta^2}{4}-\frac{\theta_0^2}{4}\right)\right\}}} \tag{37}$$

Integrating both sides of the Eq. (37), provides

$$\sqrt{2}X = \int \frac{d\theta}{\sqrt{\left\{Ra\left(Mc^2+Nr+1\right)\left(\frac{\theta^3}{3}-\frac{\theta_0^3}{3}\right)-\left(Mc^2Q-H\right)\left(\theta-\theta_0\right)-\left\{Mc^2Q\gamma-\left(M^2+Nr\right)\right\}\left(\frac{\theta^2}{4}-\frac{\theta_0^2}{4}\right)\right\}}} + C_* \tag{38}$$

Where the arbitrary constant C_* is found from

$$X = 0, \quad \frac{d\theta}{dX} = 0 \quad \rightarrow \theta = \theta_e \tag{39}$$

suppose that

$$G\left(\theta; Ra, Mc, H, Q, \theta_o\right) = G\left(\theta; Ra, N, Q_h, \theta_o\right)$$
$$= \int \frac{d\theta}{\sqrt{\left\{Ra\left(Mc^2+Nr+1\right)\left(\frac{\theta^3}{3}-\frac{\theta_0^3}{3}\right)-\left(Mc^2Q-H\right)\left(\theta-\theta_0\right)-\left\{Mc^2Q\gamma-\left(M^2+Nr\right)\right\}\left(\frac{\theta^2}{4}-\frac{\theta_0^2}{4}\right)\right\}}} \tag{40}$$

where

$$N = Mc^2 + Nr, \quad Q_h = Mc^2Q - H \tag{41}$$

The integral in Eq. (40) is expressible (e.g. via Wolfram's Mathematica) in term of incomplete elliptic integrals of the first kind. For instant

$$G(\theta;1,1,1,\theta_o)=\sqrt{\frac{\alpha_1^2}{3+6\theta_o+\alpha_1}}\left\{\frac{\left[\begin{array}{l}\sqrt{\frac{3+6\theta_o+\alpha_1}{\alpha_1}}\sqrt{\frac{-3-6\theta_o+\alpha_1}{\alpha_1}}EllipticF\left(\sqrt{\frac{3+6\theta_o+\alpha_1}{2\alpha_1}},\sqrt{\frac{2\alpha_1}{3+6\theta_o+\alpha_1}}\right)\alpha_2\\ -3\sqrt{\frac{3+2\theta_o+\alpha_1+4\theta}{\alpha_1}}\sqrt{\theta_o-\theta}\sqrt{\frac{-3-2\theta_o+\alpha_1+4\theta}{\alpha_1}}EllipticF\left(\sqrt{\frac{3+2\theta_o+\alpha_1+4\theta}{2\alpha_1}},\sqrt{\frac{2\alpha_1}{3+6\theta_o+\alpha_1}}\right)\alpha_3\end{array}\right]}{\alpha_2\alpha_3}\right\} \tag{42}$$

where

$$\alpha_1=\sqrt{57-12\theta_o-12\theta_o^2}$$

$$\alpha_2=\sqrt{6\theta^3-18\theta+9\theta^2-6\theta_o^3+18\theta_o-9\theta_o^2}$$

$$\alpha_3=\sqrt{2-2\theta_o-2\theta_o^2}$$

$$EllipticF\left(\sqrt{\frac{3+6\theta_o+\alpha_1}{2\alpha_1}},\sqrt{\frac{2\alpha_1}{3+6\theta_o+\alpha_1}}\right)=\left\{\sqrt{\frac{3+6\theta_o+\alpha_1}{2\alpha_1}}+\sum_{n=1}^{\infty}\left(\prod_{n=1}^{\infty}\frac{2n-1}{2n}\right)\left(\sqrt{\frac{2\alpha_1}{3+6\theta_o+\alpha_1}}\right)^{2n}\left\{\begin{array}{l}\frac{-cos\left(\sqrt{\frac{3+6\theta_o+\alpha_1}{2\alpha_1}}\right)}{2n}\left\{\begin{array}{l}sin^{2n-1}\left(\sqrt{\frac{3+6\theta_o+\alpha_1}{2\alpha_1}}\right)\\+\sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)...(2n-2k+1)}{2^k(n-1)(n-2)...(n-k)}sin^{2n-2k-1}\left(\sqrt{\frac{3+6\theta_o+\alpha_1}{2\alpha_1}}\right)\end{array}\right\}\\+\frac{(2n-1)!!}{2^n n!}\left(\sqrt{\frac{3+6\theta_o+\alpha_1}{2\alpha_1}}\right)\end{array}\right\}\right\}$$

Therefore, the closed-form solution of Eq. (17) can be implicitly expressed as

$$X=G(\theta;Ra,N,Q_h,\theta_o) \tag{43}$$

It should be stated that the unknown θ_o in the closed-form solution is found from the following boundary condition

$$X=0,\quad \theta=1\quad \rightarrow 0=G(1;Ra,N,Q_h,\theta_o)\quad \rightarrow G(1;Ra,N,Q_h,\theta_o)=0$$

This means that for any given N, *Ra and Q*, θ_o is obtained from

$$G(1;Ra,N,Q_h,\theta_o)=0 \tag{44}$$

With the aid of Wolfram's Mathematica, the computations of the function $G\left(\theta; Ra, N, Q_h, \theta_o\right)$ are carried out.

5. Results and Discussion

The numerical solutions are simulated using MATLAB and the parametric as well as sensitivity analyses are carried out using the codes. The results of the MOL are verified with the results of the exact analytical method as presented in Tables 1. It is shown from the Table that MOL is a very convenient mathematical method for the analysis of nonlinear fin thermal models.

Table 1: Comparative of results via MOL with Exact and NUM for θ(X) when Rd = 0.5, **ε** = 0.1, Ra = 0.4, Nc = 0.3, Nr = 0.2, H= 0.1

X	Exact	MOL
0.0	0.86349907	0.86349932
0.1	0.86481693	0.86481718
0.2	0.86877611	0.86877635
0.3	0.87539328	0.87539346
0.4	0.88469639	0.88469654
0.5	0.89672597	0.89672617
0.6	0.91153054	0.91153085
0.7	0.92917693	0.92917724
0.8	0.94974112	0.94974143
0.9	0.97331368	0.97331389
1.0	1.00000000	1.00000000

The significance of various parameters of the nonlinear model on the thermal management enhancement of thermal systems using the solutions presented are graphical represented for pictorial discussion in Figures 2-6. It is shown in Figures 2 and 3 that that when the conductive-radiative and conductive-convective increased, the dimensionless fin temperature decreases which leads to increase in heat transfer rate through the fin and the thermal efficiency of the fin.

This means that the local temperature in the extended surface increases as the conduction-convection convection-radiative parameters increase. The low value of the convective-conductive and radiative-conductive parameters, *Nc* and *Nr* implies a relatively thick and short fin of very high thermal conductivity while a high value of the convective-conductive and radiative-conductive parameters indicates a relatively thin and long fin of a very low thermal conductivity. Therefore, the thermal efficiency of the fin is favoured at low values of convective-conductive and radiative-conductive parameters, i.e., a relatively thick and short fin with a high thermal conductivity.

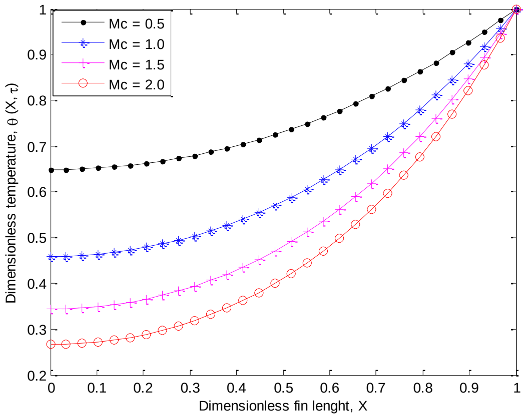

Fig. 2 Effects of convective-conductive parameter on fin temperature

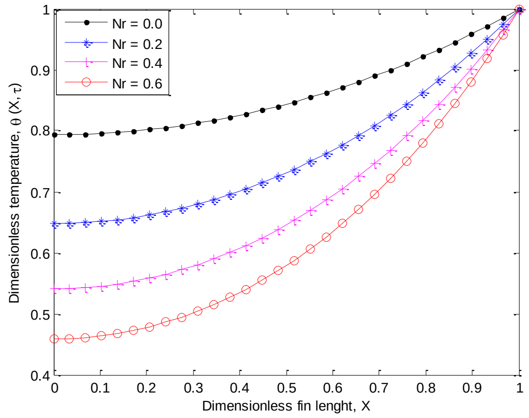

Fig. 3 Effects of radiative-conducitve parameter on fin temperature

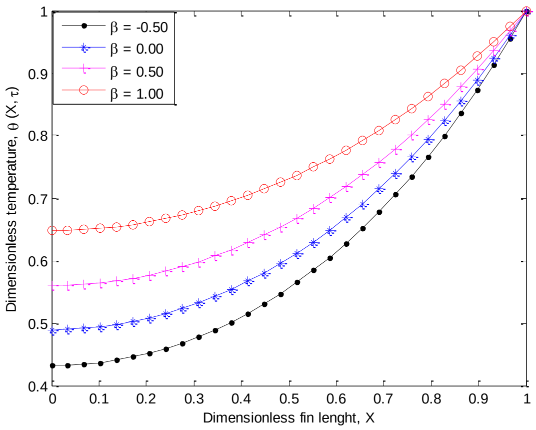

Fig. 4 Effect of the fin thermal conductivity factor on its temperature

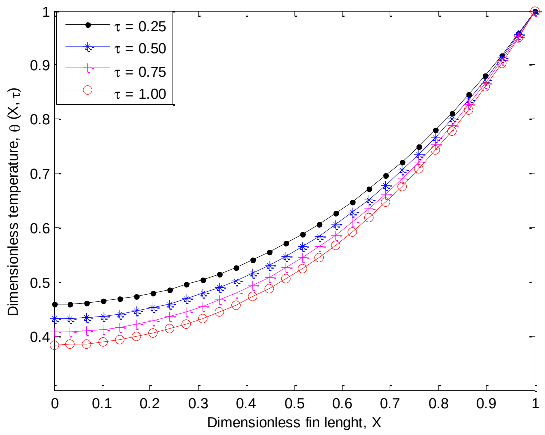

Fig. 5 Fin temperature for different time evolution

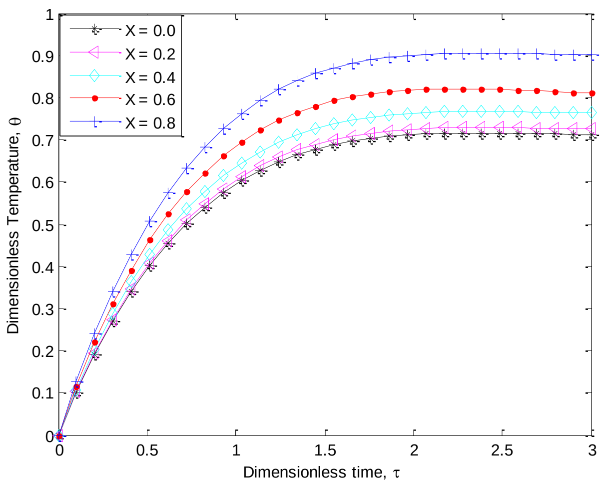

Fig. 6 Fin temperature history at various locations in the material

The thermal response of the fin to the variation of the thermal conductivity of the fin material is shown in Fig. 4. It is shown that the fin temperature is amplified as the thermal conductivity is augmented due to an increase in the fin local temperature which reduces the fin capacity to dissipate heat to the environment. The temperature profiles of the fin at different time evolutions are displayed in Fig. 5 while Fig. 6 presents the temperature histories of the extended surfaces at different positions in the material. The fin temperature increases at the different positions as the heating time progresses.

5. Conclusion

In this work, the potency of method of lines for the prediction of heat transfer characteristics of a convective-radiative solid fin with temperature-invariant thermal conductivity has been demonstrated. It has been shown that the computational method gives accurate results with high convergence and small error and without the computational burden like pure finite difference, finite element, finite volume and finite analytic methods. Therefore, the versality of the method of lines for solving linear and nonlinear PDEs is again established in this study. Also, form the solution of the method, the effect of various parameters of the nonlinear model on the thermal management enhancement of thermal systems were explored. The graphical representation of the thermal behaviour of the extended surfaces has been presented and the results have been discussed. The study has showed that when the conductive-radiative and conductive-convective parameters increased, the dimensionless fin temperature decreases which leads to increase in heat transfer rate through the fin and the thermal efficiency of the porous fin. The thermal efficiency and effectiveness of the fin is favoured at low values of conductive-radiative and conductive-convective parameters of the extended surfaces. This study will assist in proper thermal analysis of fins and in the design of passive heat enhancement devices used for thermal and electronic systems. The result of the study will help in the passive device design.

References

[1]. S. Kiwan, A. Al-Nimr. Using Porous Fins for Heat Transfer Enhancement. ASME J. Heat Transfer 2001; 123:790–5.

[2] S. Kiwan, Effect of radiative losses on the heat transfer from porous fins. Int. J. Therm. Sci. 46(2007a)., 1046-1055

[3] S. Kiwan, O. Zeitoun, Natural convection in a horizontal cylindrical annulus using porous fins. Int. J. Numer. Heat Fluid Flow 18 (5)(2008), 618-634.

[4] S. Kiwan. Thermal analysis of natural convection porous fins. Tran. Porous Media 67(2007b), 17-29.

[5] R. S. Gorla, A. Y. Bakier. Thermal analysis of natural convection and radiation in porous fins. Int. Commun. Heat Mass Transfer 38(2011), 638-645.

[6] B. Kundu, D. Bhanja, K. S. Lee. A model on the basis of analytics for computing maximum heat transfer in porous fins. Int. J. Heat Mass Transfer 55 (25-26)(2012) 7611-7622.

[7] M. T. Darvishi, R. Gorla, R.S., Khani, F., Aziz, A.-E. Thermal performance of a porus radial fin with natural convection and radiative heat losses. Thermal Science, 19(2) (2015) 669-678.

[8] A. Jordan, S. Khaldi, M. Benmouna, A. Borucki. Study of non-linear heat transfer problems, Revue Phys. Appl., 22(1987),101-105.

[9] B. Kundu, P.K. Das. Performance analysis and optimization of straight taper fins with variable heat transfer coefficient, International Journal of Heat and Mass Transfer, 45(2002), 4739-4751.

[10] F. Khani, M. A. Raji, H. H. Nejad. Analytical solutions and efficiency of the nonlinear fin problem with temperature-dependent thermal conductivity and heat transfer coefficient, Commun Nonlinear Sci Numer Simulat, 14(2009), 3327-3338.

[11] S. R. Amirkolei and D. D. Ganji D.D. Thermal performance of a trapezoidal and rectangular profiles fin with temperature-dependent heat transfer coefficient, thermal conductivity and emissivity, Indian J. Sci. Res., 1(2)(2014), 223-229.
[12] A. Aziz, M. N. Bouaziz. A least squares method for a longitudinal fin with temperature dependent internal heat generation and thermal conductivity, Energy Conversion and Management, 52(2011), 2876-2882.
[13] M. G. Sobamowo. Thermal analysis of longitudinal fin with temperature dependent properties and internal heat generation using Galerkin's method of weighted residual, Applied Thermal Engineering, 99(2016), 1316-1330.
[14] D. D. Ganji, M. Raghoshay, M. Rahimi, M. Jafari. Numerical investigation of fin efficiency and temperature distribution of conductive, convective and radiative straight fins, IJRRAS, 2010, 230-237.
[15]. M. G. Sobamowo, O. M. Kamiyo and O. A. Adeleye. Thermal performance analysis of a natural convection porous fin with temperature-dependent thermal conductivity and internal heat generation. Thermal Science and Engineering Progress. 1(2017) 39-52.
[16] M. G. Sobamowo. Thermal performance and optimum design analysis of fin with variable thermal conductivity using double decomposition method. Journal of Mechanical Engineering and Technology. 9(1)(2017), 1-32.
[17] M. G. Sobamowo, L. O. Jayesimi and J. D. Femi-Oyetoro. Heat transfer study in a convective-radiative fin with temperature-dependent thermal conductivity and magnetic field using variation of parameter method. Journal of Applied Mathematics and Computational Mechanics, 16(3)(2017), 85-96.
[18] A. Moradi and H. Ahmadikia. Analytical solution for different profiles of fin with temperature dependent thermal conductivity, Mathematical Problems in Engineering, 2010, 1-15
[19] S. Sadri, M. R. Raveshi and S. Amiri. Efficiency analysis of straight fin with variable heat transfer coefficient and thermal conductivity, Journal of Mechanical Science and Technology, 26(4)(2011), 1283-1290.
[20] P. L. Ndlovu and R. J. Moitsheki. Analytical solutions for steady heat transfer in longitudinal fins with temperature dependent properties, Mathematical Problems in Engineering, 2013, 1-14.
[21] S. Mosayebidorcheh, D. D. Ganji, M. Farzinpoor. Approximate solution of the nonlinear heat transfer equation of a fin with the power-law temperature- dependent thermal conductivity and heat transfer coefficient, Propulsion and Power Research, 3(1)(2014), 41-47.
[22] S. E. Ghasemi, M. Hatami, D. D. Ganji. Thermal analysis of convective fin with temperature-dependent thermal conductivity and heat generation, Case Studies in Thermal Engineering, 4(2014), 1-8.
[23] D.D. Ganji and A.S. Dogonchi. Analytical investigation of convective heat transfer of a longitudinal fin with temperature- dependent thermal conductivity, heat transfer coefficient and heat generation, International Journal of Physical Sciences, 9(21) (2014), 466-474.
[24]. M. G. Sobamowo, O. A. Adeleye and A. A. Yinusa. Analysis of convective-radiative porous fin with temperature-dependent internal heat generation and magnetic using homotopy perturbation method. Journal of computational and Applied Mechanics. 12(2)(2017), 127-145.
[25] C. Arslanturk. Optimization of straight fins with step change in thickness and variable thermal conductivity by homotopy perturbation method, J. of Thermal Science and Technology, 10(2010), 9-19.
[26] D. D. Ganji, Z. Z. Ganji, H. D. Ganji (2011). Determination of temperature distribution for annular fins with temperature dependent thermal conductivity by HPM, Thermal Science, 15(1)(2011), 111-115.
[27] H. A. Hoshyar, I. Rahimipetroudi, D. D. Ganji, A. R. Majidian. Thermal performance of porous fins with temperature-dependent heat generation via the homotopy perturbation method and

collocation method, Journal of Applied Mathematics and Computational Mechanics, 14(4)(2015), 53-65.
[28] K. W. Haley; J. W. Westwater. Boiling heat transfer from a single fin to a boiling liquid. Chem. Eng. Sci. 20(1965): 711
[29] F. S. Lai, Y. Y. Hsu. Temperature distribution in a fin partially cooled by nucleate boiling. AIChE J. 13(1967): 817-821
[30] A. D. Kraus AD (1988) Sixty-five years of extended surface technology. Appl. Mech. Rev. 41(1988): 321-364
[31] S. P. Liaw, R. H. Yeh. Fins with temperature dependent surface heat flux-I. Single heat transfer mode. Int. J. Heat Mass Transfer 37(1994), 1509-1514
[32] S. P. Liaw SP; R. H. Yeh. Fins with temperature dependent surface heat flux-II. Multi-boiling heat transfer, Int. J. Heat Mass Transfer 37(1994): 1517-1524
[33] H. C. Unal HC (1985) Determination of the temperature distribution in an extended surface with a non-uniform heat transfer coefficient. Int. J. Heat Mass Transfer 28(1985): 2279-2284
[34] H. C. Unal. Determination of the temperature distribution in an extended surface with a non-uniform heat transfer coefficient, Int. J. Heat Mass Transfer, 28(1986), 2279-2284.
[35] H. C. Unal. A simple method of dimensioning straight fins for nucleate pool boiling, Int. J. Heat Mass Transfer, 29(1987), 640-644, 1987.
[36] H. C. Unal. An analytic study of boiling heat transfer from a fin, Int. J. Heat Mass Transfer, 30(1987), 341-349.
[37] R. H. Yeh, S. P. Liaw. (1990) An exact solution for thermal characteristics of fins with power-law heat transfer coefficient. Int. Comm. Heat Mass Transfer 17(1990): 317 330
[38] A. K. Sen, S. Trinh. An exact solution for the rate of heat transfer from a rectangular fin governed by power law-type temperature dependence. ASME J. Heat Transfer 108(1986): 457-459
[39] S. P. Liaw, R. H. Yeh. Analysis of pool boiling heat transfer on a single cylindrical fin. J. Chin. Soc. Mech. Eng. 11(1990): 448-454
[40] R. H. Yeh, S. P. Liaw. Theoretical study of a fin subject to various types of heat transfer. Tatung Journal 20(1990), 59-66
[41]. W. W. Lin, D. J. Lee. Boiling on a straight pin fin. AIChE J. 42(1996), 2721-2728
[42] W. W. Lin, D. J. Lee. Experimental study on pin fin boiling. Heat Transfer Science and Technology, Ed. B. X. Wang, Higher Education Pub. Beijing, 1996.
[43] H. C. Unal. An analytical study of boiling heat transfer from a fin. Int. J. Heat Mass Transfer 30(1987): 341-342
[44] W. W. Lin, D. J. Lee. Boiling on a straight pin fin with variable thermal conductivity. Heat and Mass Transfer 34 (1999) 381-386.
[45] M. G. Sobamowo, S. A. Ibrahim, M. O. Salami. A Study on Thermal Performance of Palladium as Material for Passive Heat Transfer Enhancement Devices in Thermal and Electronics Systems. Semiconductor Science and Information Devices. 2(2)(2020), 15-24.
[45] M. Hatami, D. D. Ganji. Thermal performance of circular convective-radiative porous fins with different section shapes and materials. Energy Conversion and Management, 76(2013)185−193.
[46] M. Hatami, G. H. R. M. Ahangar, D. D. Ganji, K. Boubaker. Refrigeration efficiency analysis for fully wet semi-spherical porous fins. Energy Conversion and Management, 84(2014) 533−540.
[47] A. Moradi, T. Hayat and A. Alsaedi, Convective-radiative thermal analysis of triangular fins with temperature-dependent thermal conductivity by DTM. Energy Conversion and Management 77 (2014) 70–77
[48] R. Gorla, R.S., Darvishi, M. T. Khani, F. Effects of variable Thermal conductivity on natural convection and radiation in porous fins. Int. Commun. Heat Mass Transfer 38(2013), 638-645.

[49] S. Saedodin. M. ShahbabaeiThermal Analysis of Natural Convection in Porous Fins with Homotopy Perturbation Method (HPM). Arab J Sci Eng (2013) 38:2227–2231.
[50] S. E. Ghasemi, P. Valipour, M. Hatami, D. D. Ganji. Heat transfer study on solid and porous convective fins with temperature-dependent heat -generation using efficient analytical method J. Cent. South Univ. 21(2014), 4592−4598.
[51] Y. Rostamiyan, D. D. Ganji, I. R. Petroudi, and M. K. Nejad. Analytical Investigation of Nonlinear Model Arising in Heat Transfer Through the Porous Fin. Thermal Science. 18(2)(2014), 409-417.
[52] I. R. Petroudi, D. D. Ganji, A. B. Shotorban, M. K. Nejad, E. Rahimi, R. Rohollahtabar and F. Taherinia. Semi-Analytical Method for Solving Nonlinear Equation Arising in Natural Convection Porous fin. Thermal Science, 16(5) (2012), 1303-1308.
[53] S. Abbasbandy, E. Shivanian and I. Hashim. Exact analytical solution of a forced convection in porous-saturated duct. Comm. Nonlinear Sci Numer Simulat. 16(2011), 3981–3989.
[54] R. Celi, State-Space Representation of Vortex Wakes by The Method of Lines, J. of
the American Helicopter Society, 50, (2005) ,No.2, 195-205.
[55] H. Han and Z. Huang, The Direct Method of Lines for Numerical Solution of Interface Problem, Comput. Mehtods Appl. Mech. Engrg. 171, (1999) ,61-75.
[56] A. Ismail, F. Karim, G.D. Roy and M. Ashaque, Numerical Modeling of Tsunami Via The Method of Lines, Word Academy of Science, Engineering and Technology, 32, (2007) ,No.35, 177-185.
[57] J.-P. Kauthen, The Method of Lines for Parabolic Partial Integro-Differential Equations, J.of Integral Equations and Applications, 4, (1992) ,No. 1, 69-81.
[58] A. Özdes and E. N. Aksan, The Method of Lines Solution of The Korteweg-de Vries Equation for Small Times, Int. J. Contemp. Math. Sciences. 1, (2006) , No. 13, 639 – 650.
[59] S. Pamuk and A. Erdem, The Method of Lines for The Numerical Solution of a Mathematical Model for Capillary Formation: The Role of Endothelial Cells in The Capillary, Appl. Math. and Comput. 186, (2007), 831–835.
[60] M. Sadiku and C. Obiozor, A Simple Introduction to The Method of Lines, Int. J. of Elect. Engrg. Edu. 37, (2000) ,No.3, 282- 296.
[61] W. E. Schiesser, Method of Lines Solution of the Korteweg-de Vries Equation, Superconducting Super Collider Laboratory (SSCL) Preprint-443. The university Research Association, Inc., for the U.S. Department of Energy under Contract. No. DEAC35-89ER4048, 1993.
[62] W. E. Schiesser and G. W. Griffiths, A Compendium of Partial Differential Equation Models: Method of Lines Analysis with Matlab, Cambridge University Press, USA, New York, 2009.
[63] W. E. Schiesser W. E., “The numerical methods of lines”, San Diego, CA: Academic Press, 1991.
[64] B. C. Michael and H.W. Hinds. “The method of lines and the advective Equation. Simulation”, 31: 59-69, 1978.
[65] W. E. Schiesser. The Numerical Method of Lines: Integration of Partial Differential Equations”, Academic Press, San Diego, Calif., 1991. 326.
[66] K. B. Patel. A numerical solution of travelling wave that has an increasingly steep moving front using Method of Lines” International journal of education and mathematics, 2, 01-07.
[67] G. V. Patel and K. B. The Method of Lines for Solution of the One-Dimensional Second Order Wave Equation” Mathematical Sciences International Research Journal, 4, 98-103.