Skew Field on the Binomial Coefficients in Combinatorial Geometric Series

Article Info:
Article history: Received 2022-09-28 / Accepted 2022-11-14 / Available online 2022-11-23
doi: 10.18540/jcecvl8iss11pp14859-01i

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Abstract
This paper discusses a commutative group, ring, and field under addition and multiplication of the binomial coefficients in combinatorial geometric series. The combinatorial geometric series is derived from the multiple summations of geometric series. The coefficient for each term in combinatorial geometric series refers to a binomial coefficient. This idea can enable the scientific researchers to solve the real life problems.

Keywords: Computation, Binomial Coefficient, Skew Field.

1. Introduction
When the author of this article was trying to compute the multiple summations of geometric series (Annamalai, 2010; Annamalai, 2017a; Annamalai, 2017b; Annamalai, 2017c; Annamalai, 2018a; Annamalai, 2018b; Annamalai, 2018c; Annamalai, 2018d; Annamalai, 2019a; Annamalai, 2019b; and Annamalai, 2020), a new idea stimulated his mind to create a combinatorial geometric series (Annamalai, 2022a; Annamalai, 2022b; Annamalai, 2022c; Annamalai, 2022d; Annamalai, 2022e; Annamalai, 2022f). The combinatorial geometric series is a geometric series whose coefficient of each term of the geometric series denotes the binomial coefficient \( V^n_r \). In this paper, a commutative group, ring, and field under addition of the binomial coefficients of combinatorial geometric series (Annamalai, 2022d; and Annamalai, 2022e) are introduced in detail.

2. Combinatorial Geometric Series
The combinatorial geometric series (Annamalai, 2022f; Annamalai, 2022g; Annamalai, 2002h) is derived from the multiple summations of geometric series. The coefficient of each term in the combinatorial geometric series refers to the binomial coefficient \( V^n_r \) (Annamalai, 2022i; Annamalai, 2022j; Annamalai & Siqueira, 2022b).

\[
\sum_{i_1=0}^{n} \sum_{i_2=i_1}^{n} \sum_{i_3=i_2}^{n} \cdots \sum_{i_r=i_{r-1}}^{n} x^{i_1 r} = \sum_{i=0}^{n} V^n_r x^i \quad \text{and} \quad V^n_r = \frac{(n+1)(n+2)(n+3) \cdots (n+r-1)(n+r)}{r!},
\]

where \( n \geq 0, r \geq 1 \) and \( n, r \in N = \{1, 2, 3, \ldots\} \).

Here, \( \sum_{i=0}^{n} V^n_r x^i \) refers to the combinatorial geometric series and
$V^r_n$ is the binomial coefficient for combinatorial geometric series.

$V^1_0 = 1; \ V^1_1 = 2; \ V^2_0 = 3; \ V^3_1 = 4; \ V^4_1 = 5; \ V^5_1 = 6; \cdots$

$N = \{V^0_0, V^1_0, V^2_0, V^1_1, V^1_2, \cdots \}$ is a set of natural numbers (Annamalai & Siqueira, 2022a).

$W = \{0, V^0_0, V^1_1, V^1_2, V^2_0, V^1_1, V^1_2, \cdots \}$ is a set of whole numbers (Annamalai & Siqueira, 2022a).

$Z = \{\cdots, -V^1_2, -V^1_1, -V^1_0, 0, V^1_1, V^1_2, \cdots \}$ is a set of integers.

{$+, -, \times, \div, \cdots$} is a set of binary operators, where the symbol $+$ is used for addition, the symbol $-$ for subtraction, the symbol $\times$ for multiplication, the symbol $\div$ for division, etc.

**Theorem 4.4:** $V^n_r - V^{n-r}_r = V^n_r - V^{n-r}_r, (n \geq r \& n, r \in N ).$

**Proof.** $V^n_r - V^{n-r}_r = \frac{(r+1)(r+2) \cdots (r+n-r)(r+n-r+1) \cdots (r+n-r+r)}{n!} - \frac{(r+1)(r+2)(n-3) \cdots (r+n-r)}{(n-r)!} \frac{(n+1)(n+2) \cdots (n+r-1)(n+r)}{(n-r+1)(n-r+2) \cdots (n-r+r-1)}.

Here, $r! V^n_r = (n+1)(n+2) \cdots (n+r)$ and $r! V^{n-r}_r = (n-r+1)(n-r+2) \cdots (n-r+r).$

$\therefore \ V^n_r - V^{n-r}_r = V^{n-1}_r \frac{r! V^n_r - 1}{r! V^{n-r}_r} = V^{n-1}_r \frac{V^n_r - 1}{V^{n-r}_r} = V^n_r - V^{n-r}_r.$

Hence, theorem is proved.

### 3. Ring and Field

$Z = \{-V^n_0, 0, V^n_r | n \geq 1, r \geq 0 \& n, r \in N \}$ is a set of integers.

Closure property: Addition of any two binomial coefficients is also a binomial coefficient.

$(V^n_m + V^q_p) \in Z$ for all $V^n_m, V^q_p \in Z.$

Associativity: For all $V^n_m, V^q_p, V^s_r \in Z,$ $V^n_m + (V^q_p + V^s_r) = (V^n_m + V^q_p) + V^s_r.$

Identity element: $0 + V^n_r = V^n_r + 0 = V^n_r,$ where $0$ is an additive identity.

Inversive element: $V^n_r + (-V^n_r) = (-V^n_r) + V^n_r = 0,$ where $-V^n_r$ is an additive inverse.

Commutativity: $V^n_m + V^q_p = V^q_p + V^n_m$ for all $V^n_m, V^q_p \in Z.$

$(Z, +)$ is a commutative group under addition (Annalalai & Siqueira, 2022a).

A **RING** is a non-empty set $R$ which is CLOSED under two binary operators $+$ and $\times$ and satisfying the following properties:

(1) $R$ is a commutative group under $+.$

(2) $R$ is an associativity of $\times.$ For $a, b, c \in R,$ $a \times (b \times c) = (a \times b) \times c.$

(3) $R$ has distributive properties, i.e. for all $a, b, c \in R$ the following identities hold:

$a \times (b + c) = (a \times b) + (a \times c)$ and $(b + c) \times a = (b \times a) + (c \times a).$

$\therefore (Z, +, \times)$ is a **RING.**

Note that $(Z, +, \times)$ is a **Ring with Unity** which has $1$ as multiplicative identity such that $1 \times V^n_r = V^n_r \times 1 = V^n_r$ and also Commutative Ring: $V^n_m \times V^q_p = V^q_p \times V^n_m.$

A **FIELD** is a non-empty set $F$ which is CLOSED under two binary operators $+$ and $\times$ and satisfying the following properties:

(1) $F$ is an abelian group under $+.$

(2) $F - \{0\}$ is an abelian group under $\times.$
$(Z, +, \times)$ is a **FIELD**.

Note. A division ring is a ring in which $0 \neq 1$ and every nonzero element has a multiplicative inverse. A noncommutative division ring is called a skew field. A commutative division ring is called a field.

4. **Conclusion**

In this article, a commutative group, ring, and field were formed on the binomial coefficients of combinatorial geometric series under addition and multiplication. This new idea can enable the scientific researchers for research and development further (Annamalai, 2010).

**References**


