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Vector Space on the Binomial Coefficients in Combinatorial Geometric Series

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Abstract
A vector space is a group of objects that is closed under finite vector addition and scalar multiplication. This paper discusses a vector space under addition and multiplication of binomial coefficients defined in combinatorial geometric series. The combinatorial geometric series is a geometric series with binomial coefficients that is derived from the multiple summations of geometric series. This idea can enable the scientific researchers to solve the real world problems.
Keywords: computation, binomial coefficient, vector space

1. Introduction
When the author of this article was trying to compute the multiple summations of geometric series (Annamalai, et al., 2010, 2017a, 2017b, 2017c, 2018a, 2018b, 2018c, 2018d, 2019, 2020), a new idea stimulated his mind to create a combinatorial geometric series (Annamalai, et al., 2022a, 2022b, 2022c, 2022d, 2022e, 2022f). The combinatorial geometric series is a geometric series whose coefficient of each term of the geometric series denotes the binomial coefficient V_n^r. In this paper, a commutative group, ring, field, and vector space under addition of the binomial coefficients (Fowler, 1996) of combinatorial geometric series (Annamalai, et al., 2022d, 2022e) are introduced in detail.

2. Combinatorial Geometric Series
The combinatorial geometric series (Annamalai, et al., 2022f, 2022g, 2022) is derived from the multiple summations of geometric series. The coefficient of each term in the combinatorial geometric series refers to the binomial coefficient V_n^r (Annamalai, *et al.*, 2022h, 2022i).

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n}\cdots\sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i \;\&\; V_n^r = \frac{(n+1)(n+2)(n+3)\cdots(n+r-1)(n+r)}{r!},$$

where $n \geq 0, r \geq 1 \;\; and \;\; n, r \in N = \{1, 2, 3, \cdots\}$.

Here, $\sum_{i=0}^{n} V_i^r x^i$ refers to the combinatorial geometric series and
V_n^r *is* the binomial coefficient for combinatorial geometric series.

$V_0^1 = 1;\ V_1^1 = 2;\ V_2^1 = 3;\ V_3^1 = 4;\ V_4^1 = 5;\ V_5^1 = 6;\ \cdots$

$N = \{V_0^1, V_1^1, V_2^1, V_3^1,\ V_4^1, V_4^1, \cdots\}$ is a set of natural numbers (Annamalai & Siqueira, 2022a).

Let $V_n^r = V_{Q-1}^1$, where $Q = V_n^r = \dfrac{(n+1)(n+2)(n+3)\cdots(n+r)}{r!}$.

$$V_{Q-1}^1 = \frac{(Q-1+1)}{1!} = Q \quad (OR) \quad V_{V_n^r-1}^1 = \frac{(V_n^r - 1 + 1)}{1!} = V_n^r.$$

$\therefore$ V_n^r belongs to the set of natural numbers, $i.e.\ V_n^r \in N$.

$W = \{0, V_0^1, V_1^1, V_2^1, V_3^1,\ V_4^1, V_4^1, \cdots\}$ is a set of whole numbers (Annamalai & Siqueira, 2022a).

$Z = \{\cdots, -V_2^1, -V_1^1, -V_0^1, 0, V_0^1, V_1^1, V_2^1, \cdots\}$ is a set of integers.

$\{+, -, \times, \div, \cdots\}$ is a set of binary operators, where the symbol + is used for addition, the symbol – for subtraction, the symbol × for multiplication, the symbol ÷ for division, etc.

3. Abelian Group, Ring, Field, and Vector Space

Z = $\{-V_r^n, 0, V_r^n \mid n \geq 1, r \geq 0\ \&\ n, r \in N\}$ is a set of integers.

Addition of any two binomial coefficients is also a binomial coefficient.

$\left(V_m^n + V_p^q\right) \in Z$ for all $V_m^n, V_p^q \in Z$.

Associativity: For all $V_m^n, V_p^q, V_r^s \in Z,\ V_m^n + (V_p^q + V_r^s) = (V_m^n + V_p^q) + V_r^s$.

Identity element: $0 + V_r^n = V_r^n + 0 = V_r^n$, where 0 is an additive identity .

Inverse element: $V_r^n + (-V_r^n) = (-V_r^n) + V_r^n = 0$, where $-V_r^n$ is an additive inverse.

Commutativity: $V_m^n + V_p^q = V_p^q + V_m^n$ for $all\ V_m^n, V_p^q \in Z$.

(Z, +) is an Abelian group under addition (Annamalai & Siqueira, 2022a, 2022b).

RING is a non-empty set R which is CLOSED under two binary operators + and × and satisfying the following axioms:

(1) R is an Abelian group under +.

(2) R is an associativity of ×. For a, b, c ∈ R, a × (b × c) = (a × b) × c.

(3) R has distributivity, i.e. for all a, b, c ∈ R the following identities hold:

Left-distributivity: a × (b + c) = (a × b) + (a × c).

Right-distributivity: (b + c) × a = (b × a) + (c × a).

∴ (Z, +, ×) is a **RING** (Annamalai & Siqueira, 2022a, 2022b).

Note that (Z, +, ×) is a **Ring with Unity** which has 1 as multiplicative identity such that $1 \times V_r^n = V_r^n \times 1 = V_r^n$ and also Commutative Ring: $V_m^n \times V_p^q = V_p^q \times V_m^n$.

FIELD is a non-empty set F which is CLOSED under two binary operators + and × and satisfying the following axioms:

(1) F is an abelian group under +.

(2) F – {0} is an abelian group under ×.

∴ (Z, +, ×) is a **FIELD** (Annamalai & Siqueira, 2022a, 2022b, 2022c).

Note. A division ring is a ring in which $0 \neq 1$ and every nonzero element has a multiplicative inverse. A noncommutative division ring is called a skew field. A commutative division ring is called a field.

VECTOR SPACE (Sezer & Atagün, 2016) is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the following axioms. The axioms must hold for all a, b, c in V and for scalars α and β.

1. a + b is in V.
2. a + b = b + a.
3. a + (b + c) = (a + b) + c.
4. There is a vector (called zero vector) 0 in V such that a + 0 = a.
5. For each a in V, there is a vector –a in V satisfying a + (–a) = 0.
6. αa is in V.
7. α(a + b) = αa + αb.
8. (α + β)a = αa + βa.
9. (αβ)a=α(βa).
10. 1a = a.

Let q be a real or complex number, i.e. $qV_i^r = \mu_i$ for $i = 1, 2, 3, \cdots$. Then, μ_i is a real or complex number.

Let $L = \{\mu \mid -\infty < \mu < \infty\}$ be a set of real numbers. Then L with scalars α and β is a vector space. Also, if L is a set of complex number, then L with scalars α and β is a vector space.

Theorem 3.1: Let n ≥ 0 be an integer and $\boldsymbol{P_n}$ be the set of polynomials of degree at most n ≥ 0. Members of $\boldsymbol{P_n}$ have the form

$$\sum_{i=0}^{n} qV_i^r x^i = \sum_{i=0}^{n} \mu_i x^i, (rV_i^r = \mu_i \text{ for } i = 0, 1, 2, 3, \cdots), \quad \text{that is,} \quad \mathbf{p}(x) = \sum_{i=0}^{n} \mu_i x^i.$$

where μ_i, (i = 0, 1, 2, 3, . . .), are real numbers and x is a real variable. The nonempty set $\boldsymbol{P_n}$ is a vector space.

Proof. Let us prove this theorem using the *axioms* of vector space.

Axiom 1 (**A1**):
The polynomial $\mathbf{p} + \boldsymbol{q}$ is defined as follows:

$$\boldsymbol{p}(x) + \boldsymbol{q}(x) = (\boldsymbol{p} + \boldsymbol{q})x \Rightarrow (\mu_0 + \mu_1 x + \mu_2 x^2 + \cdots + \mu_n x^n) + (\delta_0 + \delta_1 x + \delta_2 x^2 + \cdots + \delta_n x^n)$$
$$= (\mu_0 + \delta_0) + (\mu_1 + \delta_1)x + (\mu_2 + \delta_2)x^2 + \cdots + (\mu_n + \delta_n)x^n$$

which is a polynomial of degree at most $n \geq 0$.

Thus, $\boldsymbol{p} + \boldsymbol{q} \in \boldsymbol{P_n}$.

Axiom 4 (**A4**):
$\mathbf{0} = 0 + 0x + 0x^2 + 0x^3 + \cdots + 0x^n, (\textit{So zero vector is in } \boldsymbol{P_n}).$

$$(\boldsymbol{p}+\boldsymbol{0})(x) = \boldsymbol{p}(x) + \boldsymbol{0} = (\mu_0 + 0) + (\mu_1 + 0)x + (\mu_2 + 0)x^2 + \cdots + (\mu_n + 0)x^n$$
$$= \mu_0 + \mu_1 x + \mu_2 x^2 + \cdots + nx^n = p(x) \text{ and so } \boldsymbol{p} + \boldsymbol{0} = \boldsymbol{p}.$$

Axiom 6 (**A6**):
Let s be scalar. Then $(s\boldsymbol{p})(x) = s\mathbf{p}(x)$

$$= (s\mu_0) + (s\mu_1)x + (s\mu_2)x^2 + \cdots + (s\mu_n)x^n which\ is\ in\ \boldsymbol{P_n}.$$

Note that the other 7 axioms (Axioms **A1**, **A4** and **A6** were formed above) also exist *in* $\boldsymbol{P_n}$.

Hence, $\boldsymbol{p_n}$ is a vector space.

Note that $\boldsymbol{P_n}$ with complex numbers in the form of $\mu_a + i\mu_b$ is a vector space.

4. Conclusion

In this article, a commutative group, ring, field, and vector space were formed on the binomial coefficients of combinatorial geometric series under addition and multiplication. This idea can enable the scientific researchers to solve the real world problems.

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