

On the Solutions of Nonlinear Algebraic and Transcendental Equations using Adomian Decomposition Method

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Gbeminiyi Musibau Sobamowo

ORCID: <https://orcid.org/0000-0003-2402-1423>

Department of Mechanical Engineering, Faculty of Engineering, University of Lagos, Akoka
Department of Mathematics, Faculty of Science, University of Lagos, Akoka

Lagos, Nigeria

E-mail: gsobamowo@unilag.edu.ng

Antonio Marcos de Oliveira Siqueira

ORCID: <https://orcid.org/0000-0001-9334-0394>

Federal University of Viçosa, Brazil

E-mail: antonio.siqueira@ufv.br

Abstract

This work presents approximate analytical solutions of nonlinear algebraic and transcendental equations through direct applications of Adomian decomposition method. The reliability and efficiency of the method in solving transcendental and nonlinear algebraic equations are demonstrated through different illustrative numerical examples. The method is shown to be conceptually and computationally simple and straightforward without any ambiguity. Also, the method as applied in this study shows that it does not require the development of any other iterative scheme that could be used to find the solution to the nonlinear equations. Additionally, the method does not require finding symbolic or numerical derivatives of any given function. With the use of the Adomian decomposition method, there is no search for an auxiliary parameter for adjusting and controlling the rate and region of convergence of the solution. The method does not require finding the correct fixed point and it is free from the problem of choosing an appropriate initial approximation. Therefore, it is hoped that the present work will assist in providing accurate solutions to many practical problems in science and engineering.

Keywords: Nonlinear algebraic equations, Nonlinear transcendental equations; Approximate analytical solutions; Root-finding; Adomian decomposition method.

1. Introduction

Finding roots of nonlinear algebraic and transcendental equations has been among the oldest problems of mathematical methods and an important aspect in providing solutions to many practical problems in science and engineering. However, it is generally difficult to establish a general analytical method that provides exact solutions to nonlinear algebraic equations. Nevertheless, there have been several submissions such as Cardan's method, Viète's, algebraic geometry, Ferrari's method, Descartes's method, Euler's method, Lagrange resolvent, etc. in the developments of exact solutions to polynomial equations.

In fact, the past centuries have witnessed the establishments of various exact analytical solutions to quadratic, cubic and quartic equations [1-8]. However, in the early 19th century, Abel and Galois ingeniously and rigorously demonstrated in their impossibility theorem that there exists no general formula for zeroes of a polynomial equation of degree five or higher [9]. Therefore, the general quintic equation and other higher-order polynomials cannot be solved algebraically in terms of a finite number of additions, subtractions, multiplications, divisions, and root extractions. Such Abel's impossibility theorem puts a period to longstanding search for a 'magic' formula for

polynomials of higher orders. However, in recent times, Waston's and Dummit's methods were developed to establish analytical solutions to quintic equations while Buya's method and Hagedorn's method have been submitted for solving sextic or hexic equations. Abel-Ruffin theorem and Kulkarni's method were put forward to solve octic equations while another Kulkarni's method was also developed to provide analytical solutions for nonic equations. De' Moivre theorem can be used to solve polynomial equation of any power but of a reduced form. Nonetheless, all these methods only provide analytical solutions to polynomial equations.

At the other hand, transcendental equations and other nonlinear algebraic equations have been solved analytically with the aid of Lambert W-function. However, the method of Lambert W-function is used to solve the nonlinear equations in which the unknown appears both outside and inside an exponential function or a logarithm. Consequently, in the continuous quest of finding roots of nonlinear algebraic equations, there have been continuous applications of numerical methods as the viable options, even with all their inherent limitations.

However, one major gap in literature is that Abel's impossibility theorem did not state whether polynomial equations can be solved with infinite power series or not. Therefore, in recent years, an infinite power series approach such as Adomian perturbation method (ADM) has been used to find the roots of nonlinear algebraic equations and more, importantly to solve nonlinear differential equations [10-20].

With the aid of the ADM, the approximate solution of the nonlinear equation is considered as an infinite series converging to the accurate solution. However, such power series solution involves determination of Adomian polynomials which increases the computational effort and time. Its slow rate of convergence for problems of wide region or domain is a great shortcoming of the method.

Homotopy perturbation method have been applied to develop some iterative methods to solve nonlinear algebraic equations [21]–[24]. Some other computational schemes have been developed [25-42] to solve nonlinear algebraic equations. A critical review of the developed methods in previous studies point to the fact that they numeric in nature which means that they are based on iterative or numerical procedures and on the idea of successive approximations that start with one or more initial approximations to the required roots. Also, many of the methods in the review works require symbolic derivatives which might be difficult to get in some complicated functions.

The limitations of the analytical and approximate analytical methods often result to make recourse to numerical methods in finding roots to polynomials, transcendental, and other nonlinear algebraic equations. In such mathematical adventures, bisection method is one of the oldest root-finding numerical methods. Although, the method is simple and its convergence is guaranteed, it is generally slow and works when the root to be estimated is of even-order. Choosing an initial guess or estimate close to root has no advantage in the application of the method and it may lead to carrying out many iterations to converge. Using method of regular falsi which seems to be an improvement over the bisection method, but unfortunately, the method yields an estimate without useable known error bound. Interestingly, the applications of linear fixed-point iteration method to find root of a nonlinear algebraic equation comes with an increased rate of convergence over bisection method and method of regular falsi.

However, sometimes, selection of the correct fixed-point or iteration function poses serious challenges in the use of the numerical method. In fact, in fixed-point iteration method, convergence can be slow or non-existence. Newton-Raphson method is taken as the most popular root-finding numerical method with high rate of convergence. Unfortunately, the convergence of the Newton-Raphson method to the required solution is not guaranteed i.e., sometimes, for a given equation and for a given initial guess or estimate, one may not get the required solution. The method converges provided the initial approximation is chosen sufficiently close to the root, otherwise, the procedure may lead to an endless cycle. This shows that the method is very sensitive and grossly dependent on initial guess or starting values. Such an initial guess may be too far from the local root, and it

may give a zero derivative and loop indefinitely. In fact, the Newton-Raphson method has poor global convergence properties. It converges slowly near local maxima and local minima, due to oscillation. Furthermore, such slow convergence is witnessed when Newton-Raphson is used for a problem with multiple roots.

Moreover, the method encounters problem when the value of the inherent derivative is very small or zero as such can lead to division by zero problem and inflection point issue can occur. Furthermore, root jumping might take place thereby not getting intended solution. Newton-Raphson method requires symbolic derivatives which might be difficult or virtually impossible to get especially for some complicated functions. Secant method has been used to obviate the derivative zero problem in Newton-Raphson method as the method does not require the derivatives of the given function. Although, the method is taken as one of the most economical numerical methods that give rapid convergence at a low cost, it requires two initial guesses or estimates for starting and it can produce erratic results when the approximations become close together. In fact, most often in numerical methods, choosing the right initial estimate(s), developing derivative(s) and the finding the correct fixed-point pose serious difficulties in the computational methods.

Motivated by the limitations and the gap in the past works and to the best of the author's knowledge, it can be stated that Adomian decomposition method has not been directly applied to solve nonlinear algebraic and transcendental equations, especially when the given equation does not have a linear term. Therefore, in this study, it is demonstrated that the direct applications of Adomian decomposition method is not only limited to solve nonlinear differential and integral equations but also, it is capable of solving nonlinear algebraic equations. Several numerical examples are given to show the reliability, performance and efficiency of the method in solving nonlinear algebraic equations.

2. Adomian Decomposition method

Adomian decomposition method (ADM) is also a non-perturbative method for solving linear and nonlinear differential equations [43-44]. Because of its obvious advantages in many areas, ADM has fast gained ground [45-54]. ADM provides highly accurate and excellent approximations symbolic solutions to non-linear equations.

2.1 2.4. Principle of Adomian decomposition method

The principle of the method is described as follows. The general nonlinear equation is in the form

$$Lu + Ru + Nu = g \quad (1)$$

The linear terms are decomposed into $L + R$, with L taken as the highest order linear derivative which is easily invertible and R as the remainder of the linear operator of less order than L . where g is the system input or the source term and u is the system output, Nu represents the nonlinear terms, which is assumed to be analytic. L^{-1} is regarded as the inverse operator of L and is defined by a definite integration from 0 to x , i.e.

$$[L^{-1} f](x) = \int_0^x f(v)dv \quad (2)$$

If L is a second-order operator, then L^{-1} is a two fold indefinite integral i.e. L^{-1} could be expressed as

$$[L^{-1} f](x) = \int_1^x \int_0^x f(v)dv dv \quad (3)$$

Applying the inverse operator L^{-1} to the both sides of Eq. (17), and using the given conditions, the resulting equation could be written as

$$u = \mu(x) - L^{-1}Ru - L^{-1}Nu \tag{4}$$

Where $\mu(x) = \lambda_x + L^{-1}g$ and λ_x represents the term arising from integrating the source term $g(x)$.

The Adomian methods decomposes the solution $u(x)$ into a series

$$u = \sum_{m=0}^{\infty} u_m \tag{5}$$

and the nonlinear term into a series

$$Nu = \sum_{m=0}^{\infty} A_m \tag{6}$$

Where A_m 's are Adomian's polynomials of u_0, u_1, \dots, u_m and are obtained for the nonlinearity $Nu = f(u)$ from the recursive formula

$$A_m = \frac{1}{m!} \left[\frac{d^m}{d\zeta^m} [fu(\zeta)] \right]_{\zeta=0} = \frac{1}{m!} \left[\frac{d^m}{d\zeta^m} f \left(\sum_{i=0}^{\infty} \zeta^i y_i \right) \right]_{\zeta=0} \quad m = 0, 1, 2, 3, \dots \tag{7}$$

Where ζ is a grouping parameter of convenience.

The Adomian decomposition method defines the solution of the function $f(x)$ to be approximated as

$$f(x) = \sum_{m=0}^{\infty} f_m(x) \tag{8}$$

3. Numerical Examples

To demonstrate the simplicity, reliability and efficiency of the direct homotopy perturbation method in solving nonlinear algebraic equations, the following polynomial and transcendental equations are clearly solved as presented under this section as presented as follows:

Example 3.1: Find the roots of the following quadratic equation using Adomian decomposition method

$$x^2 + 4x + 3 = 0. \tag{9}$$

The above equation can be expressed as

$$x = -\frac{3}{4} - \frac{x^2}{4} \tag{10} \quad \text{With}$$

the application of Adomian decomposition method,

$$\sum_{n=0}^{\infty} x = -\frac{3}{4} - \frac{1}{4} \sum_{n=0}^{\infty} x^2 \quad (11)$$

Expressing the nonlinear term as an infinite series of Adomian polynomials, we have

$$\sum_{n=0}^{\infty} x = -\frac{3}{4} - \frac{1}{4} \sum_{n=0}^{\infty} A_n \quad (12)$$

After expansion

$$x_0 + x_1 + x_2 + x_3 + \dots = -\frac{3}{4} - \frac{1}{4} (A_0 + A_1 + A_2 + A_3 + \dots) \quad (13)$$

Therefore, we can write that

$$x_0 = -\frac{3}{4},$$

$$x_1 = -\frac{1}{4} A_0 = -\frac{1}{4} x_0^2$$

$$x_2 = -\frac{1}{4} A_1 = -\frac{1}{4} (2x_0 x_1),$$

$$x_3 = -\frac{1}{4} A_2 = -\frac{1}{4} (x_1^2 + 2x_0 x_2),$$

$$x_4 = -\frac{1}{4} A_3 = -\frac{1}{4} (2x_0 x_3 + 2x_1 x_2), \quad (14)$$

$$x_5 = -\frac{1}{4} A_4 = -\frac{1}{4} (x_2^2 + 2x_1 x_3 + 2x_0 x_4),$$

$$x_6 = -\frac{1}{4} A_5 = -\frac{1}{4} (2x_0 x_5 + 2x_1 x_4 + 2x_2 x_3),$$

$$x_7 = -\frac{1}{4} A_6 = -\frac{1}{4} (x_3^2 + 2x_0 x_6 + 2x_1 x_5 + 2x_2 x_4),$$

$$x_8 = -\frac{1}{4} A_7 = -\frac{1}{4} (2x_0 x_7 + 2x_1 x_6 + 2x_2 x_5 + 2x_3 x_4),$$

$$x_9 = -\frac{1}{4} A_8 = -\frac{1}{4} (x_4^2 + 2x_0 x_8 + 2x_1 x_7 + 2x_2 x_6 + 2x_3 x_5),$$

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On solving the above equations, we have

$$\begin{aligned}
 x_0 &= -0.75, \quad x_1 = -0.140625, \quad x_2 = -0.052734, \quad x_3 = -0.024719, \quad x_4 = -0.012977, \\
 x_5 &= -0.007300, \quad x_6 = -0.004302, \quad x_7 = -0.002621, \quad x_8 = -0.001634, \quad x_9 = -0.001043
 \end{aligned}
 \tag{15}$$

From the definition of Adomian decomposition method, therefore

$$x = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + \dots
 \tag{16}$$

From Eqs. (15) and (16), we have

$$\begin{aligned}
 x &= -0.75 - 0.140625 - 0.052734 - 0.024719 - 0.012977 - 0.007300 \\
 &\quad - 0.004302 - 0.002621 - 0.001634 - 0.001043 + \dots = -0.997955
 \end{aligned}
 \tag{17}$$

Table 1: Solution of the Equation of Example 3.1

Number of Iteration (n)	n-term Solution	Absolute Error
1	-0.750000	0.250000
2	-0.890625	0.109375
3	-0.943359	0.056641
4	-0.968078	0.031922
5	-0.981055	0.018945
6	-0.988355	0.011645
7	-0.992657	0.007343
8	-0.995278	0.004722
9	-0.996912	0.003088
10	-0.997995	0.002045

The exact solutions for the roots of the equation are -1 and -3. The above solution shows that the results of the ADM is approaching a negative root of -1, which is one of the roots of the equation. It should be stated that the rate of convergence can be accelerated using Shank transformation.

Example 3.2: Find the roots of the following cubic equation using Adomian decomposition method

$$x^3 - 5x + 3 = 0.
 \tag{18}$$

The above equation can be expressed as

$$x = \frac{3}{5} + \frac{x^3}{5}
 \tag{19}$$

With the application of Adomian decomposition method,

$$\sum_{n=0}^{\infty} x = \frac{3}{5} + \frac{1}{5} \sum_{n=0}^{\infty} x^3
 \tag{20}$$

Expressing the nonlinear term as an infinite series of Adomian polynomials, we have

$$\sum_{n=0}^{\infty} x = \frac{3}{5} + \frac{1}{5} \sum_{n=0}^{\infty} A_n \quad (21)$$

After expansion

$$x_0 + x_1 + x_2 + x_3 + \dots = \frac{3}{5} + \frac{1}{5} (A_0 + A_1 + A_2 + A_3 + \dots) \quad (22)$$

Therefore, we have

$$x_0 = \frac{3}{5},$$

$$x_1 = \frac{1}{5} A_0 = \frac{1}{5} x_0^3,$$

$$x_2 = \frac{1}{5} A_1 = \frac{1}{5} (3x_0^2 x_1),$$

$$x_3 = \frac{1}{5} A_2 = \frac{1}{5} (3x_0^2 x_2 + 3x_1^2 x_0),$$

$$x_4 = \frac{1}{5} A_3 = \frac{1}{5} (3x_0^2 x_3 + x_1^3 + 6x_0 x_1 x_2),$$

$$x_5 = \frac{1}{5} A_4 = \frac{1}{5} (3x_0^2 x_4 + 3x_0 x_2^2 + 6x_0 x_1 x_3 + 3x_1^2 x_2),$$

$$x_6 = \frac{1}{5} A_5 = \frac{1}{5} (3x_0^2 x_5 + 6x_0 x_2 x_3 + 6x_0 x_1 x_4 + 3x_1 x_2^2 + 3x_1^2 x_3),$$

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The solutions of the above equations are

$$\begin{aligned} x_0 &= 0.600000, & x_1 &= 0.0432000, & x_2 &= 0.0093312, & x_3 &= 0.0026874, \\ x_4 &= 0.0008868, & x_5 &= 0.0003169, & x_6 &= 0.0001194 \end{aligned} \quad (23)$$

From the basic principle of ADM,

$$x = 0.600000 + 0.0432000 + 0.0093312 + 0.0026874 + 0.0008868 + 0.0003169 + 0.0001194 + \dots = 0.6565417$$

Table 2: Solution of the Equation of Example 3.2

Number of Iteration (n)	n-term Solution	Absolute Error
1	0.6000000	0.0566204
2	0.6432000	0.0134204
3	0.6525312	0.0040892
4	0.6552186	0.0014018
5	0.6561054	0.0005150
6	0.6564223	0.0001981
7	0.6565417	0.0000787

The exact solutions of the roots of the given equation are -2.49086362, 0.6566204, 1.83424318. Table 2 shows that the solution through the ADM is approaching the lowest positive root of 0.6566204.

Example 3.3: Determine the root of the following cubic equation with the aid of Adomian decomposition method

$$x^3 - 7x^2 + 14x - 6 = 0. \quad (24)$$

The above equation can be expressed as

$$x = \frac{3}{7} - \frac{x^3}{14} + \frac{x^2}{2} \quad (25)$$

With the application of Adomian decomposition method,

$$\sum_{n=0}^{\infty} x = \frac{3}{7} - \frac{1}{14} \sum_{n=0}^{\infty} x^3 + \frac{1}{2} \sum_{n=0}^{\infty} x^2 \quad (26)$$

Expressing the nonlinear term as an infinite series of Adomian polynomials, we have

$$\sum_{n=0}^{\infty} x = \frac{3}{7} - \frac{1}{14} \sum_{n=0}^{\infty} A_n + \frac{1}{2} \sum_{n=0}^{\infty} B_n \quad (27)$$

After expansion

$$x_0 + x_1 + x_2 + x_3 + \dots = \frac{3}{7} - \frac{1}{14} (A_0 + A_1 + A_2 + A_3 + \dots) + \frac{1}{2} (B_0 + B_1 + B_2 + B_3 + \dots) \quad (28)$$

Therefore, we have

$$x_0 = \frac{3}{7},$$

$$x_1 = -\frac{1}{14} A_0 + \frac{1}{2} B_0 = -\frac{1}{14} x_0^3 + \frac{1}{2} x_0^2,$$

$$x_2 = -\frac{1}{14}A_1 + \frac{1}{2}B_1 = -\frac{1}{14}(3x_0^2x_1) + \frac{1}{2}(2x_0x_1),$$

$$x_3 = -\frac{1}{14}A_2 + \frac{1}{2}B_2 = -\frac{1}{14}(3x_0^2x_2 + 3x_1^2x_0) + \frac{1}{2}(x_1^2 + 2x_0x_2) = 0,$$

$$x_4 = -\frac{1}{14}A_3 + \frac{1}{2}B_3 = -\frac{1}{14}(3x_0^2x_3 + x_1^3 + 6x_0x_1x_2) + \frac{1}{2}(2x_0x_3 + 2x_1x_2),$$

$$x_5 = -\frac{1}{14}A_4 + \frac{1}{2}B_4 = -\frac{1}{14}(3x_0^2x_4 + 3x_0x_2^2 + 6x_0x_1x_3 + 3x_1^2x_2) + \frac{1}{2}(x_2^2 + 2x_1x_3 + 2x_0x_4),$$

$$x_6 = -\frac{1}{14}A_5 + \frac{1}{2}B_5 = -\frac{1}{14}(3x_0^2x_5 + 6x_0x_2x_3 + 6x_0x_1x_4 + 3x_1x_2^2 + 3x_1^2x_3) + \frac{1}{2}(2x_0x_5 + 2x_1x_4 + 2x_2x_3),$$

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The solution of the above equations are

$$\begin{aligned} x_0 &= 0.4285714, \quad x_1 = 0.0862141, \quad x_2 = 0.0335563, \quad x_3 = 0.0160944, \\ x_4 &= 0.0080918, \quad x_5 = 0.0046883, \quad x_6 = 0.0029661 \end{aligned} \tag{29}$$

Therefore, we have

$$\begin{aligned} x &= 0.4285714 + 0.0862141 + 0.0335563 + 0.0160944 \\ &+ 0.0080918 + 0.0046883 + 0.0029661 + \dots = 0.5801824 \end{aligned} \tag{30}$$

Table 3: Solution of the Equation of Example 3.3

Number of Iteration (n)	n-term Solution	Absolute Error
1	0.4285714	0.1572150
2	0.5147855	0.0710009
3	0.5483418	0.0374446
4	0.5644362	0.0213502
5	0.5725280	0.0132584
6	0.5772163	0.0085701
7	0.5801824	0.0056040

The exact solutions of the roots of the given equation are 0.5857864, 3.000000 and 3.4142135. The solution of the equation using ADM is approaching the lowest positive root of 0.5857864 as the absolute error is approaching zero.

Example 3.4: Solve the following cubic equation using Adomian decomposition method

$$x^3 + 4x^2 + 8x + 8 = 0. \quad (31)$$

The above equation can be expressed as

$$x = -1 - \frac{x^3}{8} - \frac{x^2}{2} \quad (32)$$

With the application of Adomian decomposition method,

$$\sum_{n=0}^{\infty} x = -1 - \frac{1}{8} \sum_{n=0}^{\infty} x^3 - \frac{1}{2} \sum_{n=0}^{\infty} x^2 \quad (33)$$

Expressing the nonlinear term as an infinite series of Adomian polynomials, we have

$$\sum_{n=0}^{\infty} x = -1 - \frac{1}{8} \sum_{n=0}^{\infty} A_n - \frac{1}{2} \sum_{n=0}^{\infty} B_n \quad (34)$$

After expansion

$$x_0 + x_1 + x_2 + x_3 + \dots = -1 - \frac{1}{8}(A_0 + A_1 + A_2 + A_3 + \dots) - \frac{1}{2}(B_0 + B_1 + B_2 + B_3 + \dots) \quad (35)$$

Therefore, we have

$$x_0 = -1,$$

$$x_1 = -\frac{1}{8}A_0 - \frac{1}{2}B_0 = -\frac{1}{8}x_0^3 - \frac{1}{2}x_0^2,$$

$$x_2 = -\frac{1}{8}A_1 - \frac{1}{2}B_1 = -\frac{1}{8}(3x_0^2x_1) - \frac{1}{2}(2x_0x_1),$$

$$x_3 = -\frac{1}{8}A_2 - \frac{1}{2}B_2 = -\frac{1}{8}(3x_0^2x_2 + 3x_1^2x_0) - \frac{1}{2}(x_1^2 + 2x_0x_2),$$

$$x_4 = -\frac{1}{8}A_3 - \frac{1}{2}B_3 = -\frac{1}{8}(3x_0^2x_3 + x_1^3 + 6x_0x_1x_2) - \frac{1}{2}(2x_0x_3 + 2x_1x_2),$$

$$x_5 = -\frac{1}{8}A_4 - \frac{1}{2}B_4 = -\frac{1}{8}(3x_0^2x_4 + 3x_0x_2^2 + 6x_0x_1x_3 + 3x_1^2x_2) - \frac{1}{2}(x_2^2 + 2x_1x_3 + 2x_0x_4),$$

$$x_6 = -\frac{1}{8}A_5 - \frac{1}{2}B_5 = -\frac{1}{8}(3x_0^2x_5 + 6x_0x_2x_3 + 6x_0x_1x_4 + 3x_1x_2^2 + 3x_1^2x_3) - \frac{1}{2}(2x_0x_5 + 2x_1x_4 + 2x_2x_3),$$

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When the above equations are solved, we have

$$\begin{aligned} x_0 &= -1.0000000, \quad x_1 = -0.37500000, \quad x_2 = -0.23437500, \quad x_3 = -0.1640625, \\ x_4 &= -0.1179199, \quad x_5 = -0.0835876, \end{aligned} \tag{36}$$

Therefore, we have

$$x = (-1.0000000) + (-0.3750000) + (-0.2343750) + (-0.1640625) + (-0.1179199) + (-0.0835876) + \dots = -1.974945$$

Table 4: Solution of the Equation of Example 3.4

Number of Iteration (n)	n-term Solution	Absolute Error
1	-1.000000	1.000000
2	-1.375000	0.625000
3	-1.609375	0.3906250
4	-1.7734375	0.2265625
5	-1.8913574	0.1086426
6	-1.9749450	0.0250550

The exact solutions of the roots of the given equation are -2.00000 , $-1 + \sqrt{3}i$ and $-1 - \sqrt{3}i$. The result of the ADM is approaching the real root of -2.0000 as the absolute error is reducing to 0. The rate of convergence can be accelerated using Shank transformation.

Example 3.5: Solve the following quintic equation using Adomian decomposition method

$$x^5 - 3x^4 + 2x^3 + 5x^2 - 6x - 4 = 0. \tag{37}$$

Eq. (4) can be rearranged as

$$x = -\frac{2}{3} + \frac{x^5}{6} - \frac{x^4}{2} + \frac{x^3}{3} + \frac{5x^2}{6} \tag{38}$$

With the application of Adomian decomposition method,

$$\sum_{n=0}^{\infty} x = -\frac{2}{3} + \frac{1}{6} \sum_{n=0}^{\infty} x^5 - \frac{1}{2} \sum_{n=0}^{\infty} x^4 + \frac{1}{3} \sum_{n=0}^{\infty} x^3 + \frac{5}{6} \sum_{n=0}^{\infty} x^2 \tag{39}$$

Expressing the nonlinear term as an infinite series of Adomian polynomials, we have

$$\sum_{n=0}^{\infty} x = -\frac{2}{3} + \frac{1}{6} \sum_{n=0}^{\infty} A_n - \frac{1}{2} \sum_{n=0}^{\infty} B_n + \frac{1}{3} \sum_{n=0}^{\infty} C_n + \frac{5}{6} \sum_{n=0}^{\infty} D_n \tag{40}$$

After expansion

$$\begin{aligned} x_0 + x_1 + x_2 + x_3 + \dots &= -\frac{2}{3} + \frac{1}{6}(A_0 + A_1 + A_2 + A_3 + \dots) - \frac{1}{2}(B_0 + B_1 + B_2 + B_3 + \dots) \\ &\quad + \frac{1}{3}(C_0 + C_1 + C_2 + C_3 + \dots) + \frac{5}{6}(D_0 + D_1 + D_2 + D_3 + \dots) \end{aligned} \tag{41}$$

Therefore, we have

$$x_0 = -\frac{2}{3},$$

$$x_1 = \frac{1}{6}A_0 - \frac{1}{2}B_0 + \frac{1}{3}C_0 + \frac{5}{6}D_0 = \frac{1}{6}x_0^5 - \frac{1}{2}x_0^4 + \frac{1}{3}x_0^3 + \frac{5}{6}x_0^2,$$

$$x_2 = \frac{1}{6}A_1 - \frac{1}{2}B_1 + \frac{1}{3}C_1 + \frac{5}{6}D_1 = \frac{1}{6}(5x_0^4x_1) - \frac{1}{2}(4x_0^3x_1) + \frac{1}{3}(3x_0^2x_1) + \frac{5}{6}(2x_0x_1),$$

$$x_3 = \frac{1}{6}A_2 - \frac{1}{2}B_2 + \frac{1}{3}C_2 + \frac{5}{6}D_2 = \frac{1}{6}(10x_0^3x_1^2 + 5x_0^4x_2) - \frac{1}{2}(4x_0^3x_2 + 6x_0^2x_1^2) + \frac{1}{3}(3x_0^2x_2 + 3x_1^2x_0) + \frac{5}{6}(x_1^2 + 2x_0x_2),$$

$$x_4 = \frac{1}{6}A_3 - \frac{1}{2}B_3 + \frac{1}{3}C_3 + \frac{5}{6}D_3 = \frac{1}{6}(10x_0^2x_1^3 + 20x_0^3x_1x_2 + 5x_0^4x_3) - \frac{1}{2}(4x_0^3x_3 + 12x_0^2x_1x_2 + 4x_0x_1^3) + \frac{1}{3}(3x_0^2x_3 + x_1^3 + 6x_0x_1x_2) + \frac{5}{6}(2x_0x_3 + 2x_1x_2),$$

$$x_5 = \frac{1}{6}A_4 - \frac{1}{2}B_4 + \frac{1}{3}C_4 + \frac{5}{6}D_4 = \frac{1}{6}(5x_0x_1^4 + 30x_0^2x_1^2x_2 + 10x_0^3x_2^2 + 20x_0^3x_1x_3 + 5x_0^4x_4) - \frac{1}{2}(4x_0^3x_4 + 12x_0^2x_1x_3 + 6x_0^2x_2^2 + 12x_0x_1^2x_2 + x_1^4) + \frac{1}{3}(3x_0^2x_4 + 3x_0x_2^2 + 6x_0x_1x_3 + 3x_1^2x_2) + \frac{5}{6}(x_2^2 + 2x_1x_3 + 2x_0x_4),$$

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The solutions of the above equations give

$$x_0 = -0.66666667, \quad x_1 = 0.15089163, \quad x_2 = 0.01366097, \quad x_3 = -0.03656980, \tag{42}$$

Therefore, we have

$$x = (-0.66666667) + (0.15089163) + (0.01366097) + (-0.03656980) + \dots = -0.53868386 \tag{43}$$

Table 5: Solution of the Equation of Example 3.5

Number of Iteration (n)	n-term Solution	Absolute Error
1	-0.66666667	0.137780621
2	-0.51577504	0.013111009
3	-0.50211407	0.026771979
4	-0.53868387	0.009797821

The exact solutions of the roots of the given equation are -0.528886049, -1.09890396, 1.76518196. It can be seen that the above solution shows that the scheme is approaching the real root of -0.528886049 as the absolute error is approaching 0.

Example 3.6: Solve the following nonlinear algebraic simultaneous equations using Adomian decomposition method

$$x^2 - 10x + y^2 + 8 = 0. \tag{44}$$

$$xy^2 + x - 10y + 8 = 0 \tag{45}$$

The above equation can be expressed as

$$x = \frac{4}{5} + \frac{x^2}{10} + \frac{y^2}{10} \tag{46}$$

$$y = \frac{4}{5} + \frac{x}{10} + \frac{xy^2}{10} \tag{47}$$

With the application of Adomian decomposition method,

$$\sum_{n=0}^{\infty} x = \frac{4}{5} + \frac{1}{10} \sum_{n=0}^{\infty} x^2 + \frac{1}{10} \sum_{n=0}^{\infty} y^2 \tag{48}$$

$$\sum_{n=0}^{\infty} y = \frac{4}{5} + \frac{1}{10} \sum_{n=0}^{\infty} x + \frac{1}{10} \left(\sum_{n=0}^{\infty} xy^2 \right) \tag{49}$$

Expressing the nonlinear term as an infinite series of Adomian polynomials, we have

$$\sum_{n=0}^{\infty} x = \frac{4}{5} + \frac{1}{10} \sum_{n=0}^{\infty} A_n + \frac{1}{10} \sum_{n=0}^{\infty} B_n \tag{50}$$

$$\sum_{n=0}^{\infty} y = \frac{4}{5} + \frac{1}{10} \sum_{n=0}^{\infty} x + \frac{1}{10} \sum_{n=0}^{\infty} C_n \tag{51}$$

After expansion

$$x_0 + x_1 + x_2 + x_3 + \dots = \frac{4}{5} + \frac{1}{10}(A_0 + A_1 + A_2 + A_3 + \dots) + \frac{1}{10}(B_0 + B_1 + B_2 + B_3 + \dots) \quad (52)$$

$$y_0 + y_1 + y_2 + y_3 + \dots = \frac{4}{5} + \frac{1}{10}(x_0 + x_1 + x_2 + x_3 + \dots) + \frac{1}{10}(C_0 + C_1 + C_2 + C_3 + \dots) \quad (53)$$

Therefore, we have

$$x_0 = \frac{4}{5},$$

$$y_0 = \frac{4}{5}$$

$$x_1 = \frac{1}{10}A_0 + \frac{1}{10}B_0 = \frac{1}{10}x_0^2 + \frac{1}{10}y_0^2 = 0,$$

$$y_1 = \frac{1}{10}A_0 + \frac{1}{10}B_0 = \frac{1}{10}x_0 + \frac{1}{10}xy_0^2,$$

$$x_2 = \frac{1}{10}A_1 + \frac{1}{10}B_1 = \frac{1}{10}(2x_0x_1) + \frac{1}{10}(2y_0y_1)$$

$$y_2 = \frac{1}{10}x_1 + \frac{1}{10}C_1 = \frac{1}{10}x_1 + \frac{1}{10}(x_1y_0^2),$$

$$x_3 = \frac{1}{10}A_2 + \frac{1}{10}B_2 = \frac{1}{10}(x_1^2 + 2x_0x_2) + \frac{1}{10}(y_1^2 + 2y_0y_2)$$

$$y_3 = \frac{1}{10}x_2 + \frac{1}{10}C_2 = \frac{1}{10}x_2 + \frac{1}{10}(x_0y_1^2 + 2x_1y_0y_1),$$

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On solving the above equations to the sixth power of the embedding parameter p , we have

$$x = 0.997853, \quad y = 0.997562 \quad (54)$$

The exact solutions of the roots of the given equation are $x = 1$ and $y = 1$. The absolute errors in the approximate solutions are 0.002147 and 0.0024380. It can be seen that the above solution shows that the scheme is approaching the roots of the equations as the absolute errors approach zero.

Example 3.7: Find the roots of the following transcendental equation using Adomian decomposition method

$$x - 2 - e^{-x} = 0. \quad (55)$$

In order to apply Adomian decomposition method, the equation is expressed as

$$x = 2 + e^{-x} \quad (56)$$

With the application of Adomian decomposition method,

$$\sum_{n=0}^{\infty} x = 2 + \sum_{n=0}^{\infty} e^{-x} \quad (57)$$

Expressing the nonlinear term as an infinite series of Adomian polynomials, we have

$$\sum_{n=0}^{\infty} x = 2 + \sum_{n=0}^{\infty} A_n \quad (58)$$

After expansion

$$x_0 + x_1 + x_2 + x_3 + \dots = 2 + A_0 + A_1 + A_2 + A_3 + \dots \quad (59)$$

Therefore, we have

$$x_0 = 2,$$

$$x_1 = A_0 = e^{-x_0},$$

$$x_2 = A_1 = x_1 e^{-x_0},$$

$$x_3 = A_2 = x_2 e^{-x_0} - \frac{x_1^2}{2!} e^{-x_0},$$

$$x_4 = A_3 = x_3 e^{-x_0} - x_1 x_2 e^{-x_0} + \frac{x_1^3}{3!} e^{-x_0},$$

$$x_5 = A_4 = x_4 e^{-x_0} - \frac{(2x_1 x_3 + x_2^2)}{2!} e^{-x_0} + \frac{3x_1^2 x_2}{3!} e^{-x_0} - \frac{x_1^4}{4!} e^{-x_0},$$

$$x_6 = A_5 = x_5 e^{-x_0} - \frac{(2x_2 x_3 + 2x_1 x_4)}{2!} e^{-x_0} + \frac{(3x_1^2 x_3 + 3x_1 x_2^2)}{3!} e^{-x_0} - \frac{4x_1^3 x_2}{4!} e^{-x_0} + \frac{x_1^5}{5!} e^{-x_0},$$

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On solving the above equations, we have

$$x_0 = 2.00000000000, x_1 = 0.13533528323366, x_2 = -0.01831563889, x_3 = 0.003718128265, \\ x_4 = -0.0008945670078, x_5 = 0.0002364579676, x_6 = -0.0000663574935, x_7 = 0.0000194104211 \\ x_8 = -0.0000058532581, x_9 = 0.0000018066598,$$

Therefore, we have

$$x = 2.00000000000 + 0.13533528323366 - 0.01831563889 + 0.003718128265 \\ - 0.0008945670078 + 0.0002364579676 - 0.0000663574935 + 0.0000194104211 \\ - 0.0000058532581 + 0.0000018066598 + \dots = 2.1200286699020 \quad (60)$$

Table 7: Solution of the Equation in Example 3.7

Number of Iteration (<i>n</i>)	<i>n</i> -term Solution	Absolute Error
1	2.000000000000000	0.1200282389876
2	2.13533528323366	0.0153070442500
3	2.11701964434790	0.0038594640000
4	2.12073777261290	0.0007095336253
5	2.11984320560510	0.0001850338250
6	2.12007966357270	0.0000514245851
7	2.12001330607920	0.0000149329084
8	2.12003271650030	0.0000044775127
9	2.12002686324220	0.0000013757454
10	2.12002866990200	0.0000004309144

The exact solutions of the roots of the given equation are 2.1200282389876. It can be seen that the above solution shows that the scheme approaches the exact solution.

Example 3.8: Determine the root of the following transcendental equation using Adomian decomposition method

$$e^x - x^2 - 3x + 2 = 0. \quad (61)$$

The above equation can be written as

$$x = \frac{2}{3} - \frac{x^2}{3} + \frac{e^x}{3}. \quad (62)$$

With the application of Adomian decomposition method,

$$\sum_{n=0}^{\infty} x = \frac{2}{3} - \frac{1}{3} \sum_{n=0}^{\infty} x^2 + \frac{1}{3} \sum_{n=0}^{\infty} e^x \quad (63)$$

Expressing the nonlinear term as an infinite series of Adomian polynomials, we have

$$\sum_{n=0}^{\infty} x = \frac{2}{3} - \frac{1}{3} \sum_{n=0}^{\infty} A_n + \frac{1}{3} \sum_{n=0}^{\infty} B_n$$

After expansion

$$x_0 + x_1 + x_2 + x_3 + \dots = \frac{2}{3} - \frac{1}{3}(A_0 + A_1 + A_2 + A_3 + \dots) + \frac{1}{3}(B_0 + B_1 + B_2 + B_3 + \dots) \quad (64)$$

Therefore, we have

$$x_0 = \frac{2}{3},$$

$$x_1 = -\frac{1}{3}A_0 + \frac{1}{3}B_0 = -\frac{1}{3}x_0^2 + \frac{1}{3}e^{x_0},$$

$$x_2 = -\frac{1}{3}A_1 + \frac{1}{3}B_1 = -\frac{1}{3}(2x_0x_1) + \frac{1}{3}x_1e^{x_0},$$

$$x_3 = -\frac{1}{3}A_2 + \frac{1}{3}B_2 = -\frac{1}{3}(x_1^2 + 2x_0x_2) + \frac{1}{3}\left(x_2 + \frac{x_1^2}{2!}\right)e^{x_0},$$

$$x_4 = -\frac{1}{3}A_3 + \frac{1}{3}B_3 = -\frac{1}{3}(2x_0x_3 + 2x_1x_2) + \frac{1}{3}\left(x_3 + x_1x_2 + \frac{x_1^3}{3!}\right)e^{x_0},$$

$$x_5 = -\frac{1}{3}A_4 + \frac{1}{3}B_4 = -\frac{1}{3}(x_2^2 + 2x_1x_3 + 2x_0x_4) + \frac{1}{3}\left(x_4 + \frac{(2x_1x_3 + x_2^2)}{2!} + \frac{3x_1^2x_2}{3!} + \frac{x_1^4}{4!}\right)e^{x_0},$$

$$x_6 = -\frac{1}{3}A_5 + \frac{1}{3}B_5 = -\frac{1}{3}(2x_0x_5 + 2x_1x_4 + 2x_2x_3) + \frac{1}{3}\left(x_5 + \frac{(2x_2x_3 + 2x_1x_4)}{2!} + \frac{(3x_1^2x_3 + 3x_1x_2^2)}{3!} + \frac{4x_1^3x_2}{4!} + \frac{x_1^5}{5!}\right)e^{x_0},$$

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On solving the above equations, we have

$$x_0 = 0.666667, \quad x_1 = -0.501097, \quad x_2 = 0.102625, \quad x_3 = -0.01883,$$

$$x_4 = 0.016575, \quad x_5 = -0.013209, \quad x_6 = 0.007343, \quad x_7 = -0.004327$$

$$x_8 = 0.003109, \quad x_9 = -0.002240,$$

Therefore, we have

$$x = 0.666667 - 0.501097 + 0.102625 - 0.01883 + 0.016575 - 0.013209 + 0.007343 - 0.004327 + 0.003109 - 0.002240 + \dots = 0.256616 \quad (65)$$

Table 8: Solution of the Equation in Example 3.8

Number of Iteration (n)	n -term Solution	Absolute Error
1	0.666667	0.409137
2	0.165557	0.091973
3	0.268195	0.010665
4	0.249365	0.008165
5	0.265940	0.008410
6	0.252731	0.004799
7	0.260074	0.002544
8	0.255747	0.001783
9	0.258856	0.001326
10	0.256616	0.000914

The exact solutions of the roots of the given equation are 0.257530. It can be seen that the above solution shows that the scheme approaches the exact solution.

It could be seen from the above examples and from the procedures of homotopy perturbation method that a linear term must be in the equation for the ADM to operate. However, it is found that in some equations, there is no presence of a linear term. Under such scenario, the application of ADM will fail except a kind of modification is done to the given equation to include a linear term. Therefore, in the subsequent examples such problems will be handled.

In order to treat such problems, we adopt that the general nonlinear equation is in the form

$$Lu + Ru + Nu = c \quad (66)$$

The linear terms are decomposed into $L + R$, with L taken as the first linear term which is easily and R as the remainder of the linear operator apart from L . where c is the constant in the equation and u is the variable, Nu represents the nonlinear terms.

Example 3.9: Find the roots of the following cubic equation using Adomian decomposition method

$$x^3 + 4x^2 - 5 = 0. \quad (67)$$

The above equation does not contain a linear term. A linear term with a convenient coefficient can be introduced to make the equation be easily amendable to the form that homotopy perturbation can easily be applied. Therefore, we write

$$x^3 + 4x^2 + ax - ax - 5 = 0 \quad (68)$$

For the choice of “ a ”, the ratio of the constant in the given equation and that of “ a ” must be within two values where the needed root of the equation lies. Therefore, we have $0 \leq \frac{5}{a} \leq 1$. In this example, we choice $a=8$.

$$x^3 + 4x^2 + 8x - 8x - 5 = 0 \tag{69}$$

We can rewrite Eq. (82) as

$$x = \frac{5}{8} + x - \frac{x^2}{2} - \frac{x^3}{8} \tag{70}$$

With the application of Adomian decomposition method,

$$\sum_{n=0}^{\infty} x = \frac{5}{8} + \sum_{n=0}^{\infty} x - \frac{1}{2} \sum_{n=0}^{\infty} x^2 - \frac{1}{8} \sum_{n=0}^{\infty} x^3 \tag{71}$$

Expressing the nonlinear term as an infinite series of Adomian polynomials, we have

$$\sum_{n=0}^{\infty} x = \frac{5}{8} + \sum_{n=0}^{\infty} x - \frac{1}{2} \sum_{n=0}^{\infty} A_n - \frac{1}{8} \sum_{n=0}^{\infty} B_n \tag{72}$$

After expansion

$$x_0 + x_1 + x_2 + x_3 + \dots = \frac{5}{8} + x_0 + x_1 + x_2 + x_3 + \dots - \frac{1}{2}(A_0 + A_1 + A_2 + A_3 + \dots) - \frac{1}{8}(B_0 + B_1 + B_2 + B_3 + \dots) \tag{73}$$

Therefore, we have

$$x_0 = \frac{5}{8},$$

$$x_1 = x_0 - \frac{1}{2}A_0 - \frac{1}{8}B_0 = x_0 - \frac{1}{2}x_0^2 - \frac{1}{8}x_0^3,$$

$$x_2 = x_1 - \frac{1}{2}A_1 - \frac{1}{8}B_1 = x_1 - \frac{1}{2}(2x_0x_1) - \frac{1}{8}(3x_0^2x_1),$$

$$x_3 = x_2 - \frac{1}{2}A_2 - \frac{1}{8}B_2 = x_2 - \frac{1}{2}(x_1^2 + 2x_0x_2) - \frac{1}{8}(3x_0^2x_2 + 3x_1^2x_0),$$

$$x_4 = x_3 - \frac{1}{2}A_3 - \frac{1}{8}B_3 = x_3 - \frac{1}{2}(2x_0x_3 + 2x_1x_2) - \frac{1}{8}(3x_0^2x_3 + x_1^3 + 6x_0x_1x_2),$$

$$x_5 = x_4 - \frac{1}{2}A_4 - \frac{1}{8}B_4 = x_4 - \frac{1}{2}(x_2^2 + 2x_1x_3 + 2x_0x_4) - \frac{1}{8}(3x_0^2x_4 + 3x_0x_2^2 + 6x_0x_1x_3 + 3x_1^2x_2),$$

$$x_6 = x_5 - \frac{1}{2}A_5 - \frac{1}{8}B_5 = x_5 - \frac{1}{2}(2x_0x_5 + 2x_1x_4 + 2x_2x_3) - \frac{1}{8}(3x_0^2x_5 + 6x_0x_2x_3 + 6x_0x_1x_4 + 3x_1x_2^2 + 3x_1^2x_3),$$

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On solving the above equations, we have

$$x_0 = 0.6250000, x_1 = 0.3991699, x_2 = 0.0912166, x_3 = -0.0961684, \\ x_4 = -0.0834048, x_5 = 0.0257617, x_6 = 0.0951325$$

Therefore,

$$x = 0.6250000 + 0.3991699 + 0.0912166 - 0.0961684 \\ - 0.0834048 + 0.0257617 + 0.0951325 + \dots = 1.0567075 \quad (74)$$

Table 9: Solution of the Equation of Example 3.9

Number of Iteration (n)	n-term Solution	Absolute Error
1	0.6250000	0.3750000
2	1.0241699	0.0241699
3	1.1153865	0.1153865
4	1.0192181	0.0192181
5	0.9358133	0.0641867
6	0.9615750	0.0384250
7	1.0567075	0.0567075

The exact solutions of the roots of the given equation are 1.0000000, -1.381966011 and -3.618033989. It can be seen that the above solution shows that the scheme is approaching the real positive root of 1.0000 from both ends, as the absolute error tends to zero.

Example 3.10: Find the roots of the following transcendental equation using Adomian decomposition method

$$e^x + \sin x + \cos x = 3. \quad (75)$$

The above equation does not contain a linear term. A convenient linear term with a coefficient can be introduced to make the equation be easily amendable to the form that homotopy perturbation can easily be applied. Therefore, we write

$$e^x + \sin x + \cos x + ax - ax = 3 \quad (76)$$

For the choice of “a”, the ratio of the constant in the given equation and that of “a” must be within two values where the needed root of the equation lies. Therefore, we have $0 \leq \frac{3}{a} \leq 1$. In this example, we choice $a = 7$.

A convenient linear term is therefore introduced such that we can write,

$$e^x + \sin x + \cos x + 7x - 7x = 3 \quad (77)$$

Which can be expressed as

$$x = \frac{3}{7} - \frac{1}{7}(e^x + \sin x + \cos x - 7x) \tag{78}$$

With the application of Adomian decomposition method,

$$\sum_{n=0}^{\infty} x = \frac{3}{7} - \frac{1}{7} \left(\sum_{n=0}^{\infty} e^x + \sum_{n=0}^{\infty} \sin x + \sum_{n=0}^{\infty} \cos x - 7 \sum_{n=0}^{\infty} x \right) \tag{79}$$

Expressing the nonlinear term as an infinite series of Adomian polynomials, we have

$$\sum_{n=0}^{\infty} x = \frac{3}{7} - \frac{1}{7} \left(\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n + \sum_{n=0}^{\infty} C_n - 7 \sum_{n=0}^{\infty} x \right) \tag{80}$$

After expansion

$$x_0 + x_1 + x_2 + x_3 + \dots = \frac{3}{7} - \frac{1}{7} \left[(A_0 + A_1 + A_2 + A_3 + \dots) + (B_0 + B_1 + B_2 + B_3 + \dots) + (C_0 + C_1 + C_2 + C_3 + \dots) - (x_0 + x_1 + x_2 + x_3 + \dots) \right] \tag{81}$$

Therefore, we have

$$x_0 = \frac{3}{7},$$

$$x_1 = -\frac{1}{7} [A_0 + B_0 + C_0 - 7x_0] = -\frac{1}{7} (e^{x_0} + \sin x_0 + \cos x_0 - 7x_0),$$

$$x_2 = -\frac{1}{7} [A_1 + B_1 + C_1 - 7x_1] = -\frac{1}{7} (x_1 e^{x_0} + x_1 \cos x_0 - x_1 \sin x_0 - 7x_1),$$

$$x_3 = -\frac{1}{7} [A_2 + B_2 + C_2 - 7x_2] = -\frac{1}{7} \left(x_2 e^{x_0} + \frac{x_1^2}{2!} e^{x_0} - \frac{1}{2!} x_1^2 \sin x_0 + x_2 \cos x_0 - \frac{1}{2!} x_1^2 \cos x_0 - x_2 \sin x_0 - 7x_2 \right),$$

$$x_4 = -\frac{1}{7} [A_3 + B_3 + C_3 - 7x_3] = -\frac{1}{7} \left(x_3 e^{x_0} + x_1 x_2 e^{x_0} + \frac{x_1^3}{3!} e^{x_0} - \frac{1}{3!} x_1^3 \cos x_0 - x_1 x_2 \sin x_0 + x_3 \cos x_0 - \frac{1}{3!} x_1^3 \sin x_0 - x_1 x_2 \cos x_0 - x_3 \sin x_0 - 7x_3 \right),$$

$$x_5 = -\frac{1}{7} [A_4 + B_4 + C_4 - 7x_4] = -\frac{1}{7} \left(x_4 e^{x_0} + \frac{(2x_1 x_3 + x_2^2)}{2!} e^{x_0} + \frac{3x_1^2 x_2}{3!} e^{x_0} + \frac{x_1^4}{4!} e^{x_0} + \frac{1}{4!} x_1^4 \sin x_0 - \frac{3x_1^2 x_2}{3!} \cos x_0 - \frac{x_2^2}{2!} \sin x_0 - x_1 x_3 \sin x_0 + x_4 \cos x_0 + \frac{1}{4!} x_1^4 \cos x_0 + \frac{3x_1^2 x_2}{3!} \sin x_0 - \frac{x_2^2}{2!} \cos x_0 - x_1 x_3 \cos x_0 - x_4 \sin x_0 - 7x_4 \right),$$

$$x_6 = -\frac{1}{7}[A_5 + B_5 + C_5 - 7x_5] = -\frac{1}{7} \left(x_5 e^{x_0} + \frac{(2x_2 x_3 + 2x_1 x_4)}{2!} e^{x_0} + \frac{(3x_1^2 x_3 + 3x_1 x_2^2)}{3!} e^{x_0} + \frac{4x_1^3 x_2}{4!} e^{x_0} + \frac{x_1^5}{5!} e^{x_0} \right. \\ \left. + \frac{1}{5!} x_1^5 \cos x_0 + \frac{4x_1^3 x_2}{4!} \sin x_0 - \frac{6x_2 x_3}{3!} \sin x_0 - \frac{6x_1 x_4}{3!} \sin x_0 - \frac{x_1 x_2^2}{2!} \cos x_0 - \frac{x_1^2 x_3}{2!} \cos x_0 + x_5 \cos x_0 \right. \\ \left. - \frac{1}{5!} x_1^5 \sin x_0 + \frac{4x_1^3 x_2}{4!} \cos x_0 - \frac{6x_2 x_3}{3!} \cos x_0 - \frac{6x_1 x_4}{3!} \cos x_0 + \frac{x_1 x_2^2}{2!} \sin x_0 + \frac{x_1^2 x_3}{2!} \sin x_0 - x_5 \sin x_0 \right)$$

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On solving the above equations, we have

$$x_0 = 0.4285714, \quad x_1 = 0.0199721, \quad x_2 = 0.0141811, \quad x_3 = 0.0100645, \quad x_4 = 0.0071586, \quad (82)$$

Therefore

$$x = 0.4285714 + 0.0199721 + 0.0141811 + 0.0100645 + 0.0071586 + \dots = 0.4799477 \quad (83)$$

Table 10: Solution of the Equation of Example 3.10

Number of Iteration (n)	n-term Solution	Absolute Error
1	0.4284714	0.0687268
2	0.4485435	0.0486565
3	0.4627246	0.0344754
4	0.4727891	0.0244109
5	0.4799477	0.0172523

The above solution converges to 0.4972 at the tenth term of the ADM, which is a root of the equation. Therefore, $x = 0.4972$

Example 3.11: Find the roots of the following transcendental equation using Adomian decomposition method

$$Inx + \sinh x + \sin^2 x \cos^2 x = 15. \quad (84)$$

The above equation does not contain a linear term. A convenient linear term with a coefficient can be introduced to make the equation be easily amendable to the form that homotopy perturbation can easily be applied. Therefore, we write

$$Inx + \sinh x + \sin^2 x \cos^2 x + ax - ax = 15 \quad (85)$$

For the choice of “a”, the ratio of the constant in the given equation and that of “a” must be within two values where the needed root of the equation lies. Therefore, we have $3 \leq \frac{15}{a} \leq 4$. In this example, we choice $a = 5$.

$$Inx + \sinh x + \sin^2 x \cos^2 x + 5x - 5x = 15 \quad (86)$$

The homotopy perturbation of the equation is given as

$$x = 3 - \frac{1}{5} [Inx + \sinh x + \sin^2 x \cos^2 x - 5x] \tag{87}$$

With the application of Adomian decomposition method,

$$\sum_{n=0}^{\infty} x = 3 - \frac{1}{5} \left(\sum_{n=0}^{\infty} Inx + \sum_{n=0}^{\infty} \sinh x + \sum_{n=0}^{\infty} \sin^2 x \cos^2 x - 5 \sum_{n=0}^{\infty} x \right) \tag{88}$$

Expressing the nonlinear term as an infinite series of Adomian polynomials, we have

$$\sum_{n=0}^{\infty} x = 3 - \frac{1}{5} \left(\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n + \sum_{n=0}^{\infty} C_n - 5 \sum_{n=0}^{\infty} x \right) \tag{89}$$

After expansion

$$x_0 + x_1 + x_2 + x_3 + \dots = 3 - \frac{1}{5} \left[(A_0 + A_1 + A_2 + A_3 + \dots) + (B_0 + B_1 + B_2 + B_3 + \dots) + (C_0 + C_1 + C_2 + C_3 + \dots) - 5(x_0 + x_1 + x_2 + x_3 + \dots) \right] \tag{90}$$

Therefore, we have

$$x_0 = 3,$$

$$x_1 = -\frac{1}{5} [A_0 + B_0 + C_0 - 5x_0] = -\frac{1}{5} (Inx_0 + \sinh x_0 + \sin^2 x_0 \cos^2 x_0 - 5x_0),$$

$$x_2 = -\frac{1}{5} [A_1 + B_1 + C_1 - 5x_1] = \frac{1}{5} \left(\frac{x_1}{x_0} + x_1 \cosh x_0 + 2x_1 \sin x_0 \cos^3 x_0 - 2x_1 \sin^3 x_0 \cos x_0 - 5x_1 \right),$$

$$x_3 = -\frac{1}{5} [A_2 + B_2 + C_2 - 5x_2] = -\frac{1}{5} \left(\frac{x_2}{x_0} - \frac{x_1^2}{2x_0} + x_2 \cosh x_0 + \frac{x_1^2}{2!} \sinh x_0 + x_1 \cos^4 x_0 - 6x_1^2 \sin^2 x_0 \cos^2 x_0 + 2x_2 \sin x_0 \cos^3 x_0 + x_1^2 \sin^4 x_0 - 2x_2 \sin^3 x_0 \cos x_0 - 5x_2 \right),$$

$$x_4 = -\frac{1}{5} [A_3 + B_3 + C_3 - 5x_3] = -\frac{1}{5} \left(\frac{x_3}{x_0} - \frac{x_1 x_2}{x_0^2} + \frac{x_1^3}{3x_0^3} + x_3 \cosh x_0 + x_1 x_2 \sinh x_0 + \frac{x_1^3}{3!} \cosh x_0 - \frac{16}{3} x_1^3 \cos^3 x_0 \sin x_0 + 2x_1 x_2 \cos^4 x_0 + \frac{16}{3} x_1^3 \sin^3 x_0 \cos x_0 - 12x_1 x_2 \sin^2 x_0 \cos^2 x_0 + 2x_3 \sin x_0 \cos^3 x_0 + 2x_1 x_2 \sin^4 x_0 - 2x_3 \sin^3 x_0 \cos x_0 - 5x_3 \right),$$

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On solving the above equations, we have

$$x = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + \dots = 3.31705 \tag{91}$$

Table 12: Solutions of some other nonlinear equations using Adomian decomposition method

Equations	Approximate Analytical Solution
$x^3 + x - 1 = 0$	$x = 0.6823278$
$x^2 - (1-x)^5 = 0$	$x = 0.3459542$
$e^{-x} + \cos x = 0$	$x = 1.7461395$
$\cos x - x = 0$	$x = 0.7390851$
$x - 0.2 \sin x = 0.8$	$x = 0.9643339$
$\sin^2 x - x^2 + 1 = 0$	$x = 1.4044916$
$e^{-x} - 3x^2 = 0$	$x = 0.9100075$
$x^6 - 5x^5 + 3x^4 + x^3 + 2x^2 - 8x = 0.5$	$x = 0.6823278$
$\ln x + e^x - 2x^2 + 1 = 0$	$x = 0.1224248$

In the solutions of nonlinear algebraic equations, Shanks transform can be used to convert a slowly converging sequence to its rapidly converging counterpart effectively. The Shanks transform is an efficient relation that can accelerate the convergence rate of the series [48]. The Shanks transformation $Sh(U_n)$ of the sequence U_n is defined as,

$$Sh(U_n) = \frac{U_{n+1}U_{n-1} - U_n^2}{U_{n+1} - 2U_n + U_{n-1}} \quad (92)$$

Further speed-up may be achieved by successive implementation of the Shanks transformation, that is $Sh^2(U_n) = Sh(Sh(U_n))$, $Sh^3(U_n) = Sh(Sh(Sh(U_n)))$, $Sh^4(U_n) = Sh(Sh(Sh(Sh(U_n))))$ etc.

4. Conclusion

In this work, Adomian decomposition method has been directly applied to solve nonlinear algebraic and transcendental equations. The reliability and efficiency of the method in solving nonlinear algebraic equations have been demonstrated by different number of illustrative examples. The method has been shown to be conceptually and computationally simple and straightforward without any ambiguity. Also, the advantages and the superiority of the approximate analytical method over the other approximate analytical and numerical methods of finding the roots of nonlinear algebraic equations have been presented. Additionally, the following points should be noted

- i. The Adomian decomposition method has capability to find the complex root of an equation. Through a numerical of numerical experiments, it was found that if an initial term x_0 is appropriately chosen as a complex number close to the root, the ADM might converge to a complex root. It should be noted that a diverging series (for the equation with real coefficients) may indicate complex roots. For example, direct application of ADM to solve equation $x^3 - 3x - 5 = 0$, produces divergence results which indicates that the equation has complex root(s).
- ii. It is not unlikely that the direct approach of the ADM to the solution of nonlinear algebraic equations can produce diverging results for an equation that has real roots such as $x^5 - 4x^4 - 13x^3 + 46x^2 + 11x - 43 = 0$. However, with some mathematical manipulations, the direct application of ADM to such problem is possible. This will be presented in the further work on the study.
- iii. It can also be said that with the combination of matrix algebra and direct ADM, all the possible roots of a nonlinear equation can be found. This will also be presented in the subsequent paper of this study.

It is hoped that the present work will assist in providing accurate solutions to many practical problems in science and engineering.

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