JCEC
THE JOURNAL OF ENGINEERING AND EXACT SCIENCES

OPEN ACCESS

The Journal of Engineering and Exact Sciences – jCEC, Vol. 09 N. 09 (2023)
journal homepage: https://periodicos.ufv.br/jcec
eISSN: 2527-1075
ISSN: 2446-9416

An Alternative Method for the Gamma Function derived from Natural Logarithm and Pi Function

Article Info:
Article history: Received 2022-11-20 / Accepted 2022-03-22 / Available online 2023-06-26
doi: 10.18540/jcecvl9iss9pp16052-01e

Chinnaraji Annamalai
ORCID: https://orcid.org/0000-0002-0992-2584
Department of Management, Indian Institute of Technology,
Kharagpur, West Medinipur, West Bengal, Kharagpur – 721302, India
E-mail: anna@iitkgp.ac.in
Antonio Marcos de Oliveira Siqueira
ORCID: https://orcid.org/0000-0001-9334-0394
Chemical Engineering Graduate Program, Federal University of Viçosa, Brazil
E-mail: antonio.siqueira@ufv.br

Abstract:
Several professors of mathematics from the renowned universities in Australia, Canada, Europe, India, USA, etc. argue with me that the gamma function was not derived from the factorial function. For them, this paper presents the derivation of gamma function from the natural logarithm and Euler's factorial function. Also, a novel factorial theorem, which is alternative to the gamma function, is introduced in this article and it computes the accurate values of factorial for positive real numbers.
Keywords: Computation, Euler Integral, Factorial, Derivation.

1. Introduction

In general, the factorial for positive integer n, denoted by n!, is the product of all positive integers less than or equal to n. For example, $5! = 1 \times 2 \times 3 \times 4 \times 5 = 720$. Note that 0!=1. In this article, the gamma function is proved using the natural logarithm and Pi function [Annamalai, 2023a; Annamalai, 2023b; and Annamalai, 2023c], also known as Euler's factorial function. Also, a novel factorial theorem, which is alternative to the gamma function, is introduced in this article and it computes the accurate values of factorial for positive real numbers.

2. Natural Logarithm

The natural logarithm of a real or complex number is a technique to solve the exponential function and vice-versa (Annamalai, 2023d; and Annamalai, 2023e).

The natural logarithm is defined as follows:

$$\log_e u = \ln u \tag{1}$$

The properties of natural logarithm are given below:

$$\ln u^v = v \ln u \tag{2}$$

$$\ln uv = \ln u + \ln v \tag{3}$$

$$\ln \frac{u}{v} = \ln u - \ln v \tag{4}$$

$$\ln \frac{1}{u} = \ln 1 - \ln u = -\ln u \ (\because \ln 1 = 0) \tag{5}$$

$$\ln e = 1 \tag{6}$$

Let $e^m = n,$ then $\ln n = m.$ Also, $e^{\ln n} = n$ and $\ln e^m = m$ (7)

$$-\ln e^{-r} = -(-r) = r \tag{8}$$

3. Gamma Function

The gamma function is proved here using Pi function and natural logarithm. The Pi (**Π**) function is given below:

$$\boldsymbol{\Pi}(x) = x! = x(x-1)(x-2)\cdots 1, \forall\, x \in \boldsymbol{W}, \tag{9}$$

where $\boldsymbol{W}$ denotes the system of whole numbers.

Euler's factorial function, also known as Pi (**Π**) function, is the basis for gamma function (Abbas, 2023; Annamalai, 2023a; Annamalai, 2023b; Annamalai, 2023c; Annamalai, 2023d; and Annamalai, 2023e),

Swiss mathematician Leonhard Euler has defined the pi function by integral as follows (Assad, 2007; Borwein *et al.*, 1989; and Tsiganov, 2009):

$$\boldsymbol{\Pi}(x) = \int_0^1 (-\ln t)^n \, dt, \forall\, n \in \boldsymbol{W}, \tag{10}$$

where **ln** denotes the logarithm to the base of the mathematical constant e.

By substituting $t = e^{-x};\ dt = -e^{-x}dx;\ \ t = 0 \Rightarrow e^{-x} = 0 \Rightarrow x = \infty$

and $t = 1 \Rightarrow e^{-x} = 1 \Rightarrow x = 0$ in (10), we obtain:

$$\boldsymbol{\Pi}(x) = \int_\infty^0 -x^n\, e^{-x}dx = \int_0^\infty x^n\, e^{-x}dx, \qquad \forall\, n \in \boldsymbol{W}, \tag{11}$$

where $(-\ln t)^n = (-\ln e^{-x})^n = (-(-x))^n = x^n.$

By integrating (11), we obtain:

$$\boldsymbol{\Pi}(x) = [-x^n e^{-x}]_0^\infty - \int_0^\infty -nx^{n-1}\, e^{-x}dx \tag{12}$$

$$\boldsymbol{\Pi}(x) = [x^\infty e^{-\infty} - 0] - \int_0^\infty -nx^{n-1}\, e^{-x}dx \tag{13}$$

$\text{Here, } x^{\infty}e^{-\infty} \text{ is in indeterminate form. Let us apply L'Hospital's rule,}$

$$i.e. \lim_{x\to\infty} -\frac{x^n}{e^x} = \lim_{x\to\infty} -\frac{nx^{n-1}}{e^x} = \cdots = \lim_{x\to\infty} -\frac{n(n-1)\cdots 1}{e^x} = 0 \quad (14)$$

$$\mathbf{\Pi}(x) = 0 - \int_0^{\infty} -nx^{n-1}\, e^{-x} dx \quad (15)$$

$$\text{Now, } \mathbf{\Pi}(x) = n\int_0^{\infty} x^{n-1}\, e^{-x} dx \quad (16)$$

By substituting (11) in (16), we obtain:

$$\mathbf{\Pi}(x) = n\mathbf{\Pi}(x-1) \quad (17)$$

The gamma function $\Gamma(x)$ is obtained as follows.

$$\mathbf{\Pi}(x) = n\mathbf{\Pi}(x-1) \Longrightarrow \Gamma(x+1) = n\Gamma(x) \quad (18)$$

$$n\Gamma(x) = n\int_0^{\infty} x^{n-1}\, e^{-x} dx \Longrightarrow \Gamma(x) = \int_0^{\infty} x^{n-1}\, e^{-x} dx \quad (19)$$

The gamma function (Annamalai, 2023b; and Annamalai, 2023e), therefore, is derived from the Euler's factorial function (Pi function) that uses the actual factorial function.

4. Error in Calculation using Gamma Function

The gamma function (Abbas, 2023) for integer values is given below:

$$\Gamma(x+1) = x(x-1)! = x\Gamma(x) \quad (20)$$

Then,

$$\Gamma(1) = 0! = 1;\ \Gamma(2) = 1! = 1;\ \Gamma(3) = 2! = 2 \quad (21)$$

The gamma function (Davis, 1972; Sebah & Gourdon, 2002; and Artin, 2015) for real arguments or real numbers is shown below:

$$\Gamma(r) = (r-1)! = \int_0^{\infty} t^{r-1}\, e^{-t} dt \quad (22)$$

Then,

$$\Gamma\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2} - 1\right)! = \left(-\tfrac{1}{2}\right)! = \sqrt{\pi} \quad (23)$$

$$\Gamma\left(\tfrac{1}{2}\right) = \left(-\tfrac{1}{2}\right)! = \sqrt{\pi} \approx 1.77245385091 \quad (24)$$

but $\Gamma(3) > \Gamma\left(\tfrac{1}{2}\right) > \Gamma(2)$, which is a contradiction.

$$\Gamma\left(\tfrac{3}{2}\right) = \left(\tfrac{3}{2}-1\right)\left(\tfrac{3}{2}-1-1\right)! = \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)! = \tfrac{\sqrt{\pi}}{2} \quad (25)$$

So, $\Gamma\left(\frac{1}{2}\right)$ is not true and also $c$alculating the value of $\Gamma\left(\frac{3}{2}\right)$ using $\Gamma\left(\frac{1}{2}\right)$ can not be true. From these results, it is concluded that the calculated values using $\Gamma\left(\frac{1}{2}\right)$ are not true.

5. Factorial Theorem

Theorem 2.1: $(x+1)! = x! + \{(x+1)! - x!\}$, where $x \geq 0$.

Proof:
$x! + \{(x+1)! - x!\} = x! + \{x!\,(x+1) - x!\} = x! + x!\,(x) = x!\,\{1+x) = (x+1)!$, (x$\geq 0$).

Hence, theorem is proved.

Let us verify the theorem with positive real numbers.

$$(x+1)! = x! + \{(x+1)! - x!\} = x! + (0.64)\{(x+1)! - x!\} + (0.36)\{(x+1)! - x!\}$$
$$x! + (0.64+0.36)\{(x+1)! - x!\} = x! + \{(x+1)! - x!\}$$

Theorem 2.2: $y! = (y+1)! - \{(y+1)! - y!\}$, where $y \geq 1$.

Proof:
$$(y+1)! - \{(y+1)! - y!\} = y!\,(y+1) - \{y!\,(y+1) - y!\}$$
$$= y!\,(y+1) - y!\,(y) = y!\,(y+1-y) = y!, (y \geq 1)$$

Hence, theorem is proved.

Let us verify the theorem with positive real numbers.

$$y! = (y+1)! - \{(y+1)! - y!\}$$
$$= (y+1)! - (1-0.64)\{(y+1)! - y!\} - (1-0.36)\{(y+1)! - y!\}$$
$$= (y+1)! - (0.36)\{(y+1)! - y!\} - (0.64)\{(y+1)! - y!\} = (y+1)! - \{(y+1)! - y!\}$$

6. Factorial for Positive Real number

Let $z = i.f$ be a positive real number, where i is an integer part, f is the decimal or fractional part, and $z \geq 1$ The factorial function to positive real number is established from the perspective of the above theorems as follows:

$$z! = (i.f)! = i! + (0.f)\{(i+1)! - i!\}$$

Then,

$$(1.0)! = 1! + (0.0)\{(i+1)! - i!\} = 1 + 0 = 1$$

$$(4.6)! = 4! + (0.6)(5! - 4!) = 24 + (0.6)(120 - 24) = 81.6$$

The gamma function calculator computes the factorial function (4.6)! as follows:

$$\Gamma(5.6) = (5.6-1)! = (4.6)! = 61.5$$

7. Factorial for Rational and Irrational numbers

If $\frac{p}{q}, (q \neq 0) = i\frac{f}{q}$ is a common rational number, where $\frac{p}{q} \geq 1$, f < q, & i is an integer.

Then, $$\left(\frac{p}{q}\right)! = i! + \left(\frac{f}{q}\right)\{(i+1)! - i!\}.$$

And,

$$1! = \left(1\tfrac{0}{1}\right)! = 1! + \left(\tfrac{0}{1}\right)\{(1+1)! - 1!\} = 1 + 0 = 1$$

$$\left(\tfrac{14}{4}\right)! = \left(3\tfrac{2}{4}\right)! = 3! + \left(\tfrac{2}{4}\right)(4! - 3!) = 6 + 9 = 15$$

Let $\sqrt{x}$ be a irrational number, where $\sqrt{x} \geq 1$. If i is the integer part on $\sqrt{x}$.

Then, $$(\sqrt{x})! = i! + \sqrt{x}\{(i+1)! - i!\} - i\{(i+1)! - i!\}.$$

And,

$$(\sqrt{1})! = 1! + \sqrt{1}(2! - 1! - 1(2! - 1!) = 1 + 1 - 1 = 1$$

$$(\sqrt{15})! = 3! + \sqrt{15}(4! - 3!) - 3(4! - 3!)$$

$$(119)! = 10! + \sqrt{119}(11! - 10!) - 10(11! - 10!)$$

8. Conclusion

In this paper, factorial theorem for positive real numbers has been introduced and it is an alternative to the gamma function. This result can be used as an application in computing and mathematical sciences including probability and statistics.

References

Abbas, I. (2023) Preliminary approach to calculate the Gamma function without numerical integration. *Researchgate*, Pages 1-11.

Annamalai, C. (2023a) Gamma Function derived from Factorial Function based-Pi Function. *COE, Cambridge University Press*. https://dx.doi.org/10.33774/coe-2023-wv4rc.

Annamalai, C. (2023b) Gamma Function derived from Natural Logarithm and Pi Function. *COE, Cambridge University Press*. https://dx.doi.org/10.33774/coe-2023-wv4rc-v3.

Annamalai, C. (2023c) Proof of Gamma Function using Natural Logarithm and Pi Function. *COE, Cambridge University Press*. https://dx.doi.org/10.33774/coe-2023-wv4rc-v2.

Annamalai, C. (2023d) An Alternative Method for the Gamma Function derived from Natural Logarithm and Pi Function. *COE, Cambridge University Press*. https://dx.doi.org/10.33774/coe-2023-wv4rc-v4.

Annamalai, C. (2023e) Review on the Gamma Function and Error Correction. *SSRN Electronic Journal*. https://dx.doi.org/10.2139/ssrn.4397585.

Artin, E. (2015). *The gamma function*. Courier Dover Publications.

Assad, A. A. (2007). Leonhard Euler: A brief appreciation. *Networks: An International Journal*, *49*(3), 190-198.

Borwein, J. M., Borwein, P. B., & Dilcher, K. (1989). Pi, Euler numbers, and asymptotic expansions. *The American Mathematical Monthly*, *96*(8), 681-687.

Davis, P. J. (1972). Gamma function and related functions. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, 253-293.

Sebah, P., & Gourdon, X. (2002). Introduction to the gamma function. *American Journal of Scientific Research*, 2-18.

Tsiganov, A. V. (2009). Leonard Euler: addition theorems and superintegrable systems. *Regular and Chaotic Dynamics*, *14*, 389-406.