

Dirac equation solution in the light front via linear algebra and its particularities

Solução da equação de Dirac na frente da luz via álgebra linear e suas particularidades

Article Info:

Article history: Received 2022-06-07 / Accepted 2023-03-01 / Available online 2023-08-17

doi: 10.18540/jcecv19iss9pp16329-01e

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Abstract

In undergraduate and postgraduate courses, it is customary to present the Dirac equation defined in a space of four dimensions: three spatial and one temporal. This article discusses aspects of the Dirac equation (QED) on the light front. This proposal of coordinate transformations comes from Dirac who originally introduced three distinct forms of relativistic dynamics possible depending on the choice we make of the different hypersurfaces constant in time. The first he called instantaneous, the most common form, the hypersurface of which is specified by the boundary conditions set at $t = 0$. The second, known as the point form, has as its characterizing surface, a hyperboloid, described by the initial conditions in $x^\mu x_\mu = a^2$, being a one constant (chosen as the time of this system). The third relativistic form, known as the light front form, has its hypersurface tangent to the light cone; being defined by the initial conditions at $x^+ = t + z = 0$, and x^+ is the time in the light front system. The method of this work is deductive. Therefore, one obtains the solution of the Dirac equation for the Free Electron and for the positron in the coordinates in the light front with the particularity of the energy associated with the system being given by $p^- \propto 1/p^+$, and for moments $p^+ > 0$ we have the electron and $p^+ < 0$ we have the positron. The result of this is that the positive energy states in the light front $p^- > 0$ and negative $p^- < 0$ are independently described in the equation, and with additional, the problem at the limit $\lim_{p^+ \rightarrow 0} p^-$ that does not converge.

Keywords: relativity, Minkowski space, coordinate system, fermions.

Resumo

Nos cursos de graduação e pós-graduação costuma-se apresentar a equação de Dirac definida num espaço de quatro dimensões: três espaciais e uma temporal. Este artigo, aborda aspectos da equação de Dirac (QED) na Frente de Luz. Essa proposta de transformações de coordenadas vem de Dirac que originalmente introduziu três formas distintas de dinâmica relativística possíveis, dependendo da escolha que fazemos das diferentes hipersuperfícies constantes no tempo. A primeira ele chamou de forma instantânea, a mais comum, cuja hipersuperfície é especificada pelas condições de contorno definidas em $t = 0$. A segunda, conhecida como forma pontual, tem como superfície caracterizadora, um hiperboloide, descrita pelas condições iniciais em $x^\mu x_\mu = a^2$, sendo “ a ” uma constante (escolhida como o tempo desse sistema). A terceira forma relativística, conhecida como forma da frente de luz, tem sua hipersuperfície tangente ao cone de luz, sendo definido pelas condições iniciais em $x^+ = t + z = 0$, e x^+ é o tempo para o sistema da frente de luz. O método deste trabalho é dedutivo. Portanto, obtém-se a solução da equação de Dirac para o elétron livre e para o pósitron nas coordenadas da frente de luz com a particularidade da energia associada ao sistema ser dada por $p^- \propto 1/p^+$, sendo que para momentos $p^+ > 0$ temos o elétron e $p^+ < 0$ temos o pósitron. O resultado disso é que os estados de energia positivo na frente de luz $p^- > 0$ e negativo $p^- < 0$ são descritos de forma independente na equação e, com adicional, o problema no limite $\lim_{p^+ \rightarrow 0} p^-$ que não converge.

Palavras-chave: relatividade, espaço de Minkowski, sistema de coordenadas, férmions

1. Introduction

In the special theory of relativity, the idea of absolute time is replaced by a proper time for each inertial referential. Consequently, there is a four-dimensional space-time, and time is a fourth dimension. In this space, the metric can take on a Euclidean appearance, similar to the three-dimensional Euclidean metric used in ordinary space. This four-dimensional structure is called the Minkowski Universe (Mahon, 2021). Four coordinates in Minkowski space define an event and the distance between two events, i.e., the universe's interval is invariant. The interval of the universe is similar to the distance between two points in three-dimensional space.

Three forms of dynamics start from the description of the initial state of a relativistic system on any surface of space-time. In 1949, Dirac (Dirac, 1949) showed that it is possible to construct, in an alternative way to Minkowski's space. The particles propagate in space advancing in time from the hyperplane, also denoted as hypersurface at $t = 0$. Until a later instant $t > 0$, being the same one's defined instantaneous form. Which corresponds to relativistic theory with boundary conditions defined $t = 0$. The Point form consists of establishing the initial data on a branch of a hyperboloid, and the frontal form, which we will deal with in this article, is known as the light front, where its initial conditions are given in a hyperplane of Minkowski space containing the trajectory of light.

The coordinate system of the light front is very early in QED, QCD, and QCD in the perturbative (in the hadronic scale) problems of difficult calculations (Brodsky *et al.*, 1998). In the case of non-perturbative QCD, we have an example of this difficulty, which is how to understand the theory of the strong interaction of the canonical structure of QCD (Glasek *et al.*, 1993). Another example is that there are in the literature many effective models or theories that are inspired by QCD, and yet they cannot be derived directly from QCD, although they are more or less successful in describing hadronic phenomenology. Undoubtedly, discussing the problems of non-perturbative QCD is not a simple task. However, the fact that little progress has been made in the last twenty years in the study of no perturbative QCD (i.e., the explicit solution of the dynamics of quarks and gluons in strong coupling regions). This might force us to wonder if anyone is capable of extending the traditional perturbative field theory approach to the description of a strongly coupled theory (Brodsky *et al.*, 1998).

The covariant perturbative framework lacks a simple representation such as in the Schrödinger representation of quantum mechanics, which may be the obstacle to the development of a non-perturbative covariant framework (Carbonell *et al.*, 1998).

Relativistic covariance is a property necessary for a realistic description of many physical systems. Quantum field theory offers a formalism that describes interacting particles in a covariant form. Quantum field theory in the light front has been proposed as a version of quantum field theory that has advantages over the usual formulation. The main point is that the vacuum has a simpler form structure on the light front. Many results on the light front have been obtained by methods that work in quantum field theory, or by simply applying the results of ordinary quantum field theory to quantum field theory on the light front.

A simple example to show the importance of using the light front can be seen in the calculation of the amplitude of the bubble diagram. Figure 1 shows the bubble diagram that is calculated in most quantum field theory textbooks (Ligterink, 1996). The covariant amplitude is

$$F(p^2) = \frac{g^2}{2(2\pi)^4} \int d^4k \frac{1}{((p-k)^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} \quad (1)$$

The factor g occurs for each vertex of the diagram in figure 1, the momentum p is from the outer lines of the bubble, the internal momentum in the bubble is given by k , and the factor $1/2$ is due to symmetry in the bubble diagram

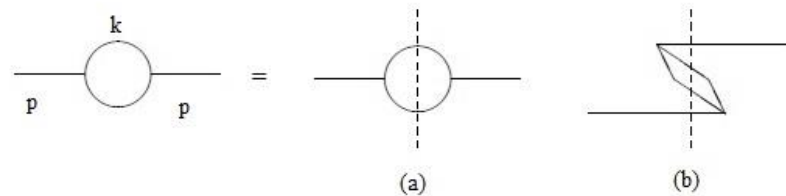


Figure 1- The bubble diagram is composed of two diagrams $t > 0$ (a) and $t < 0$ (b)

The covariant diagram shows the sum of two specific processes separated by the temporal ordering of the two vertices. We can see this if we look at the propagator, which contains two propagators; one for each pole:

$$\frac{1}{p^2 - m^2 + i\epsilon} = \frac{1}{2E_p} \left(\frac{1}{p^0 - E_p + i\epsilon} - \frac{1}{p^0 + E_p - i\epsilon} \right) \quad (2)$$

where $E_p = \sqrt{\vec{p}^2 + m^2}$.

If we separate the different poles in the scalar bubble diagram, the poles combine in only two cases for a contribution that does not vanish. One pole must be on each side of the real axis to contribute to the boundary integration. Integration over energy is equivalent to adding up all the processes for different relative times between the occurrences of the two vertices. If we integrate over the energy variable p^0 , we will have only these two contributions, corresponding to different temporal ordering of the two vertices. We refer to the two-time orderings as the diagram of the future propagator $t > 0$, figure 1 (a):

$$\frac{-ig^2}{2(2\pi)^3} \int d^3k \frac{1}{4E_{p-k}E_k} \frac{1}{(p^0 - E_{p-k} - E_k)} \quad (3)$$

In addition, the past propagator $t < 0$, figure 1 (b):

$$\frac{-ig^2}{2(2\pi)^3} \int d^3k \frac{1}{4E_{p-k}E_k} \frac{1}{(p^0 + E_{p-k} + E_k)} \quad (4)$$

respectively.

The meaning of the bubble diagram has been the subject of much discussion, figure 1. We see that it is necessary to recover a covariant amplitude. The past propagator diagram is also called a Z diagram. The contributions due to vacuum (bubble) fluctuations are due to the creation of the particle and antiparticle from the vacuum. For fermions, it also says that the vacuum is suppressed by the Pauli principle due to the presence of extra fermions. However, this does not explain the same diagram for bosons.

The past propagator diagram is absent in the light front perturbation theory. If we introduce the light front variables in (1) and integrate them under the light front energy p^- we will find only one contribution:

$$\frac{-ig^2}{2(2\pi)^3} \int_0^{p^+} dk^+ d^2k_{\perp} \frac{1}{4(p^+ - k^+)k^+} \frac{1}{\left(p^- - \frac{(p_{\perp} - k_{\perp})^2 + m^2}{2(p^+ - k^+)} - \frac{k_{\perp}^2}{2k^+} \right)}$$

We do not have the Z contribution. This is because we only have poles in a region in the complex plane when integrating via residue theory. Therefore, the light front coordinate system in this example shows us an advantage for calculations in quantum field theory. Nevertheless, it also has some problems such as the zero modes, i.e., $\lim_{p^+ \rightarrow 0} p^-$ that does not converge (Sales, *et al.*, 2001).

2. Light Front

Through the three forms of relativistic dynamics that Paul Dirac created in 1949 (Dirac, 1949), it is possible to describe the motion of its particles, depending only on the type of hyperplane chosen for the temporal evolution of the system, figure 2. Instantaneous form dynamics is the usual one, which consists of specifying the initial data (the initial position and velocity), on the surface $x^0 = ct$; the punctual form, whose surface is specified by the initial conditions in $x^{\mu}x_{\mu} = a^2$ with the constant and $x^0 > 0$. The data are calculated along a hyperboloid, and the third formulation, light front, defines the initial conditions on a three-dimensional hypersurface in space-time, forming a light front hyperplane that advances with speed close to that of light.

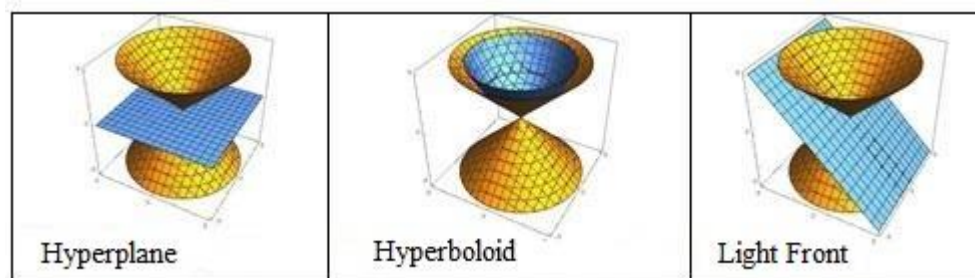


Figure 2 - Three forms of dynamics. Source (Sales *et al.*, 2020).

The coordinates of the light front are obtained, in particular, by the transformations:

$$\begin{cases} x^+ = \frac{1}{\sqrt{2}}(x^0 + x^3) \\ x^- = \frac{1}{\sqrt{2}}(x^0 - x^3), \\ \vec{x}^\perp = x^1\vec{i} + x^2\vec{j} \end{cases} \quad (5)$$

where $x^0 = ct$ is the temporal component and x^1, x^2 e x^3 are the spatial components (x, y, z) respectively. The vector \vec{x}^\perp is contained in the plan (x, y) and is transversal to the components, (x^+, x^-). The coordinate x^+ is the time parameter in front of light (Sales *et al.*, 2015) and the components (x^-, x^\perp) are associated with spatial points in front of light.

In Figure 3, we see a plane tangent to the cone of light, which, also, is tangent to the coordinate, x^- and perpendicular to coordinate x^+ ; this plan defines the light front.

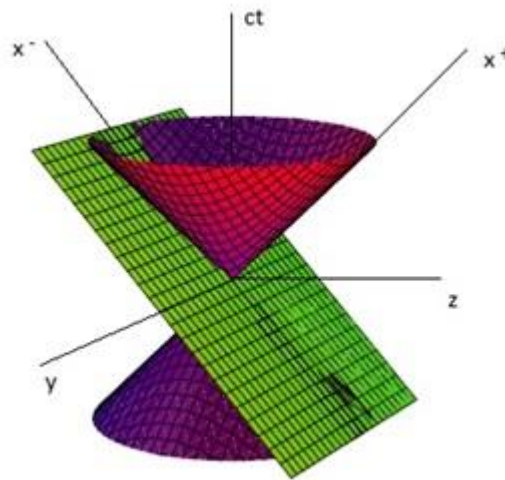


Figure 3 - Light front. Source (Sales *et al.*, 2015)

We will use the so-called Minkowski metric with the signature Bjorken-Drell (Bjorken, 1969):

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (6)$$

The inverse transformation to Minkowski space in the Bjorken-Drell metric is given by:

$$\begin{cases} x^0 = \frac{\sqrt{2}}{2}(x^+ + x^-), \\ (x^1, x^2) = \vec{x}^\perp, \\ x^3 = \frac{\sqrt{2}}{2}(x^+ - x^-) \end{cases} \quad (7)$$

where $x^0 = ct$ and $x^3 = z$.

With this, the infinitesimal distance between two points in Minkowski space-time becomes

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad (8)$$

Using (5), we obtain

$$ds^2 = 2dx^+dx^- - (dx^\perp)^2 \quad (9)$$

This induces us to have the metrics in front of the light

$$g_{LF}^{\mu\nu} = g_{LF\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \quad (10)$$

being $g_{LF}^{\mu\nu}$ the metric in the light front (Oliveira *et al.*, 2020) and using (10) we get the properties $x^+ = x_-$ and $x^\perp = -x_\perp$.

Thus, a *quadrivector* A^μ any in the light front is given by,

$$A_{FL}^\mu = (A^+, A^-, \vec{A}^\perp). \quad (11)$$

The coordinates (5) are an important quadrivector. Are the canonically conjugate moments given by

$$\begin{cases} p^+ = \frac{1}{\sqrt{2}}(p^0 + p^3), \\ \vec{p}^\perp = p^1\vec{i} + p^2\vec{j} \\ p^- = \frac{1}{\sqrt{2}}(p^0 - p^3) \end{cases} \quad (12)$$

The inverse transformation is

$$\begin{cases} p^0 = \frac{\sqrt{2}}{2}(p^+ + p^-), \\ (p^1, p^2) = \vec{p}^\perp \\ p^3 = \frac{\sqrt{2}}{2}(p^+ - p^-) \end{cases} \quad (13)$$

For the scalar product of the product between two *quadrivectors* $p_\mu x^\mu$ in the coordinates of the light front, we use (7) and (13), obtaining:

$$p_\mu x^\mu = p^-x^+ + p^+x^- - p^\perp x^\perp \quad (14)$$

The usual method to find the ratio of the energy of the massive particle associated with the coordinates in front of the light is to calculate the scalar product $p^\mu p_\mu = c^2 m_0^2$ of the *quadrivector* at these coordinates, with the rest mass m_0 . Therefore:

$$p^\mu p_\mu = p^0 - p_x^2 - p_y^2 - p_z^2 \Rightarrow 2p^+p^- - \vec{p}_\perp^2 = c^2 m_0^2,$$

therefore we have

$$p^- = \frac{c^2 m_0^2 + \vec{p}_\perp^2}{2p^+}, \quad (15)$$

where p^- is defined as the energy of the particle in the light front and the components (p^+, \vec{p}^\perp) are the moments on the light front.

We know that the energy of a free particle in Minkowski space-time is given by $p^0 = E = \pm c\sqrt{m_0^2 + \vec{p}^2}$, which shows a quadratic dependence of p^0 with \vec{p} in contrast to the linear dependence between $(p^+)^{-1}$ and p^- at the coordinates of the light front (12). Therefore, in relativistic total energy, we have two signs \pm , and in equation (12) the sign of is associated with the p^+ . Being that $p^+ \geq 0$ implies in $p^- \geq 0$ this means, on the light front the positive energy is defined by $p^- > 0$ and the negative energy by $p^- < 0$. A problem that appears here is known in the zero mode literature $p^+ = 0$. Such equality in (12) does not show a divergence to p^- . Hence, it is a limitation of this technique. In special cases, the zero mode is removed.

There are a wide variety of studies done regarding these new coordinates, the most recent being the study of the Fermion propagation model in the light front (Sales *et al.*, 2001). The fields of application range from systems with few nuclei (Brodsky *et al.*, 1998; Frederico *et al.*, 1992; Terent'ev, 1976; Kondratyuk *et al.*, 1980; Dziembowski *et al.*, 1987) to studies on the structure of light hadrons (Dziembowski *et al.*, 1987). In this coordinate, the x^+ coordinate is generally considered a parameter of time evolution due to singularity (Kim *et al.*, 1982). Other authors (Sales *et al.*, 2001; Brodsky *et al.*, 1998; Frederico *et al.*, 1992), in a framework known as the Infinite Moment, developed something related to what is studied in this work. However, it is worth informing it is not the same as the dynamics of the light front since it is not equivalent: at Infinite Moment The parameter of temporal evolution is the usual coordinate t , while we are studying the longitudinal direction of coordinate x^+ (Sales *et al.*, 2015; Suzuki *et al.*, 2015; Suzuki *et al.*, 2013).

3. Dirac Equation

Historically the Klein-Gordon equation was proposed in 1920 as a relativistic extension of the Schrödinger equation (Greiner *et al.*, 1996). The Klein-Gordon equation is not sufficient to define quantum theory completely. The obtained wave function must have an interpretation of a probability amplitude. To remedy this difficulty, Dirac proposed that the equation to be satisfied by the wave function be first-order in time, consequently, also First-Order in spatial derivatives. Thus, Dirac presented his equation for spin - $\frac{1}{2}$ fermions and, consequently,

$$(i\gamma^\mu \partial_\mu - m)\Psi(x^\mu) = 0 \tag{16}$$

where Ψ is a column matrix with four components written as

$$\Psi(x^\mu) = \begin{pmatrix} \Psi_1(x^\mu) \\ \Psi_2(x^\mu) \\ \Psi_3(x^\mu) \\ \Psi_4(x^\mu) \end{pmatrix} \tag{17}$$

Call of bi-spinor or spinor of Dirac (17). An important observation is that although $\Psi(x^\mu)$ has four components, $\Psi(x^\mu)$ is not a quadrivector. It transforms from one inertial frame to another, not by the usual Lorentz transformation. This is because the spinor is a different representation of the Lorentz group.

The matrices γ^μ in (16) are operators defined as

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (18)$$

where we have the properties $(\gamma^0)^\dagger = \gamma^0$ e $(\gamma^i)^\dagger = -\gamma^i$ com $i = 1,2,3$.

It is appropriate to observe that the matrices (17) despite having the same indices $\mu = 0,1,2,3$ of the Minkowski space-time are invariant by Lorentz transformation. This is justified because γ^μ acts on different spaces of x^μ and $\Psi(x^\mu)$.

We suppose that the solution $\Psi(x^\mu)$ of the Dirac equation for the free electron is given by a plane wave (spinor), that is:

$$\Psi(x^\mu) = u(p^\mu)e^{-ip_\mu x^\mu} \quad (19)$$

where we use the natural units, $\hbar = c = 1$ e $u(p^\mu)$ is a column matrix

$$u(p^\mu) = \begin{pmatrix} u_1(p^\mu) \\ u_2(p^\mu) \\ u_3(p^\mu) \\ u_4(p^\mu) \end{pmatrix} \quad (20)$$

Using (20) and the matrices (18) in (16), we obtain

$$\begin{pmatrix} E - m & 0 & -p_z & -(p_x - ip_y) \\ 0 & E - m & -(p_x + ip_y) & p_z \\ p_z & p_x - ip_y & -(E + m) & 0 \\ p_x + ip_y & -p_z & 0 & -(E + m) \end{pmatrix} \begin{pmatrix} u_1(p^\mu) \\ u_2(p^\mu) \\ u_3(p^\mu) \\ u_4(p^\mu) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (21)$$

The moment operators $p_1 = p_x, p_2 = p_y$ and $p_3 = p_z$ were obtained from the action of the derivative on the spinor (14), resulting $\partial_\mu e^{-ip_\mu x^\mu} = -ip_\mu e^{-ip_\mu x^\mu}$, that is:

$$(\gamma^\mu p_\mu - m)u(p^\mu) = 0 \quad (22)$$

Since the system of equations generated by this matrix product (21) is homogeneous, by Cramer's rule (Howrad, 2012), it will only have a solution other than the trivial one if the determinant of the coefficients of the variables u_i ($i = 1,2,3,4$) is null:

$$\begin{vmatrix} E - m & 0 & -p_z & -(p_x - ip_y) \\ 0 & E - m & -(p_x + ip_y) & p_z \\ p_z & p_x - ip_y & -(E + m) & 0 \\ p_x + ip_y & -p_z & 0 & -(E + m) \end{vmatrix} = 0 \quad (23)$$

Solving the determinant (23) by Laplace's rule and Sarrus ' rule (Howrad, 2012), also considering that the components of the momentum operator p commute with each other will come:

$$E = \pm\sqrt{p^2 + m^2} \quad (24)$$

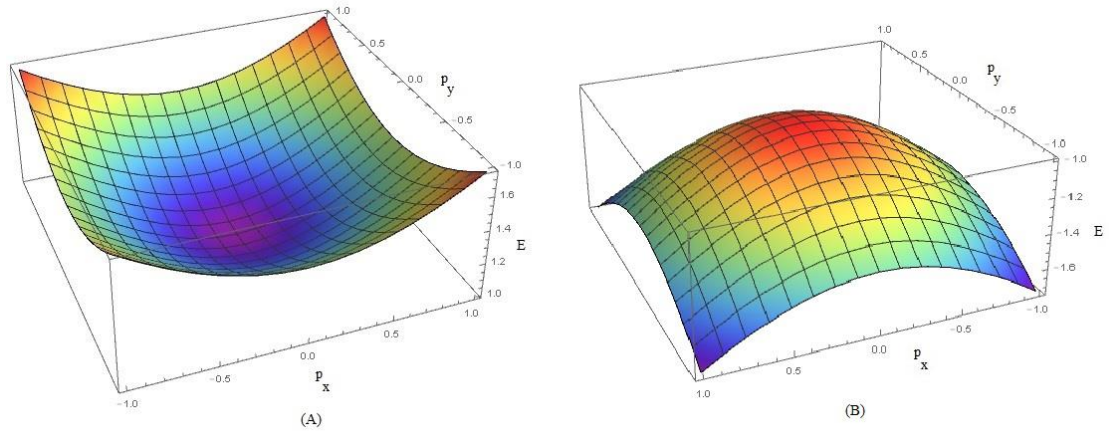


Figure 4 - Electron energy as a function of moments

Figure 4 shows the energy as a function of two moments p_x and p_y . The graph is the result of using Equation (24) for mass value $m = 1$ and moments ranging from -1 a $+1$. It is observed that the energy because it is quadratic has two values $E > 0$ and $E < 0$ in figure 4 (A) and (B) respectively.

Note that the system of equations coming from (23) is homogeneous linear and contains trivial solutions $u_1 = 0, u_2 = 0, u_3 = 0$ and $u_4 = 0$. All other solutions, if any, are called non-trivial solutions. Thus, we have the following result: a homogeneous linear system has only the trivial solution or has an infinity of solutions, with no other possibilities.

For this reason, values $E > 0$ $u_1 = 1, u_2 = 0$ for spin up, and $u_1 = 0, u_2 = 1$ for spin down are selected for simplicity. In case $E < 0$ use $u_3 = 1, u_4 = 0$ for spin up and $u_3 = 0, u_4 = 1$ for spin down. In this way, we have:

Table 1- Spinors in a frame moving in the Z direction.

$E > 0$		$E < 0$	
Spin up	Spin down	Spin up	Spin down
$u_1 = 1$	$u_1 = 0$	$u_1 = -\frac{p_z}{E + m}$	$u_1 = -\frac{p_x - ip_y}{E + m}$
$u_2 = 0$	$u_2 = 1$	$u_2 = -\frac{p_x + ip_y}{E + m}$	$u_2 = \frac{p_z}{E + m}$
$u_3 = \frac{p_z}{E + m}$	$u_3 = \frac{p_x - ip_y}{E + m}$	$u_3 = 1$	$u_3 = 0$
$u_4 = \frac{p_x + ip_y}{E + m}$	$u_4 = -\frac{p_z}{E + m}$	$u_4 = 0$	$u_4 = 1$

In Table 1 we have the values of the spins in the case of the moving frame of reference $p^\mu = (E, p_x, p_y, p_z)$ about an inertial frame at rest. For the case of the resting electron, $p^\mu = (E, 0, 0, 0)$ we have the result given in Table 2:

Table 2- Spinors in the referential at rest.

$E > 0$		$E < 0$	
Spin up	Spin down	Spin up	Spin down
$u_1 = 1$	$u_1 = 0$	$u_1 = 0$	$u_1 = 0$
$u_2 = 0$	$u_2 = 1$	$u_2 = 0$	$u_2 = 0$
$u_3 = 0$	$u_3 = 0$	$u_3 = 1$	$u_3 = 0$
$u_4 = 0$	$u_4 = 0$	$u_4 = 0$	$u_4 = 1$

As we saw above in Table 2, the Dirac equation for a free electron admits four-plane wave solutions. Two of them for positive $E > 0$, describe an electron of p momentum \vec{p} and opposite spin. The other two solutions with negative energy, $E < 0$, and opposite spin.

The possibility of the existence of continuous negative energy states of the free electron of mass m implies a paradox, Klein's paradox: an electron, in the fundamental state, can emit a photon with energy $h\nu > 2m$ and fall into the negative energy state predicted by Dirac's electron theory. Once in this negative energy state, it would continue emitting photons, since there is no minimum negative energy limit, as it extends to infinity.

Dirac solved this paradox by presenting the solution by stating that, under normal conditions, electrons, known as “Dirac’s sea”, occupied all the negative energy states.

4. Dirac Equation on the Light Front

The Dirac equation on the light front is obtained by the transformations (5) and with the scalar product (14) in (19). Therefore, the spinor (14) on the light front is then given by:

$$\Psi_{LF}(x^+, x^-, x^\perp) = u(p^+, p^-, p^\perp) e^{-i(p^-x^+ + p^+x^- - p^\perp x^\perp)} \tag{25}$$

Before we get the Dirac equation on the light front, let's rewrite (13) in matrix form

$$\begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} p^+ \\ p^1 \\ p^2 \\ p^- \end{pmatrix} \tag{26}$$

Representing the Dirac matrix (18) as

$$\gamma^\mu = \begin{pmatrix} \gamma^0 \\ \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix} \text{ and } \gamma_\mu = (\gamma_0 \ \gamma_1 \ \gamma_2 \ \gamma_3) \tag{27}$$

In equation (22) we have the product on the light front

$$\begin{aligned} \gamma_\mu p^\mu &= (\gamma_0 \quad \gamma_1 \quad \gamma_2 \quad \gamma_3) \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} p^+ \\ p^1 \\ p^2 \\ p^- \end{pmatrix} = \\ &= \begin{pmatrix} \frac{\sqrt{2}}{2}(\gamma_0 + \gamma_3) & \gamma_1 & \gamma_2 & \frac{\sqrt{2}}{2}(\gamma_0 - \gamma_3) \end{pmatrix} \begin{pmatrix} p^+ \\ p^1 \\ p^2 \\ p^- \end{pmatrix} \end{aligned}$$

The result suggests the definition

$$\gamma_{\mu_{LF}} = (\gamma_+ \quad \gamma_1 \quad \gamma_2 \quad \gamma_-)$$

or

$$\gamma_{\mu_{LF}} = \begin{pmatrix} \gamma^+ \\ \gamma^1 \\ \gamma^2 \\ \gamma^- \end{pmatrix}$$

where

$$\begin{cases} \gamma_+ = \frac{\sqrt{2}}{2}(\gamma_0 + \gamma_3) \\ \gamma_\perp = (\gamma_1, \gamma_2) \\ \gamma_- = \frac{\sqrt{2}}{2}(\gamma_0 - \gamma_3) \end{cases} \quad (28)$$

We observe in (28) the properties $\gamma_+ = \gamma^-$, $\gamma_- = \gamma^+$ e $\gamma_\perp = -\gamma^\perp$.

The transformed Dirac equation (22) in the light front is obtained, with the support of the spinor (25) and the derivatives in the light front. Thus, we have:

$$[\gamma^+ p^- + \gamma^- p^+ - \gamma^\perp p^\perp - m] u(p^+, p^-, p^\perp) = 0 \quad (29)$$

This is the Dirac equation in the light front (29), or in the most compact form:

$$\left(\gamma_{\mu_{LF}} p_{\mu_{LF}} - m \right) u_{LF} = 0$$

where $u_{LF} = u(p^+, p^-, p^\perp)$.

To find out the associated energy at the light front p^- it is necessary to use (29) in matrix form and by definition $p^+ > 0$:

$$\begin{pmatrix} \frac{p^+ + p^-}{\sqrt{2}} - m & 0 & -\left(\frac{p^+ - p^-}{\sqrt{2}}\right) & -(p_x - ip_y) \\ 0 & \frac{p^+ + p^-}{\sqrt{2}} - m & -(p_x + ip_y) & \frac{p^+ - p^-}{\sqrt{2}} \\ \frac{p^+ - p^-}{\sqrt{2}} & p_x - ip_y & -\left(\frac{p^+ + p^-}{\sqrt{2}} + m\right) & 0 \\ p_x + ip_y & -\left(\frac{p^+ - p^-}{\sqrt{2}}\right) & 0 & -\left(\frac{p^+ + p^-}{\sqrt{2}} + m\right) \end{pmatrix} \times \begin{pmatrix} u_1(p^+, p^-, p^\perp) \\ u_2(p^+, p^-, p^\perp) \\ u_3(p^+, p^-, p^\perp) \\ u_4(p^+, p^-, p^\perp) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (30)$$

In the matrix equation (30) and having the same observations in (21) and (23), the result is

$$\begin{vmatrix} \frac{p^+ + p^-}{\sqrt{2}} - m & 0 & -\left(\frac{p^+ - p^-}{\sqrt{2}}\right) & -(p_x - ip_y) \\ 0 & \frac{p^+ + p^-}{\sqrt{2}} - m & -(p_x + ip_y) & \frac{p^+ - p^-}{\sqrt{2}} \\ \frac{p^+ - p^-}{\sqrt{2}} & p_x - ip_y & -\left(\frac{p^+ + p^-}{\sqrt{2}} + m\right) & 0 \\ p_x + ip_y & -\left(\frac{p^+ - p^-}{\sqrt{2}}\right) & 0 & -\left(\frac{p^+ + p^-}{\sqrt{2}} + m\right) \end{vmatrix} = 0 \quad (31)$$

After using the same techniques in (23), we obtain

$$p^- = \frac{m^2 + p_\perp^2}{2p^+} \quad (32)$$

If we had chosen $p^+ < 0$ in (30), we would find $p^- < 0$. The result (32) is similar to (15), where we are assuming $c = 1$. The signal for p^- is arbitrary and depends on the definition of the signal of p^+ . If $p^+ \geq 0$ implies in $p^- \geq 0$, the solution (32) belongs to a positive or negative set less for $p^+ = 0$. Different from solution (24) which has the two signs for energy E. According to the article (Acevedo *et al.*, 2021) to maintain causality in the solution of the Dirac differential equation, it is important to have the two solutions for $\pm p^+$.

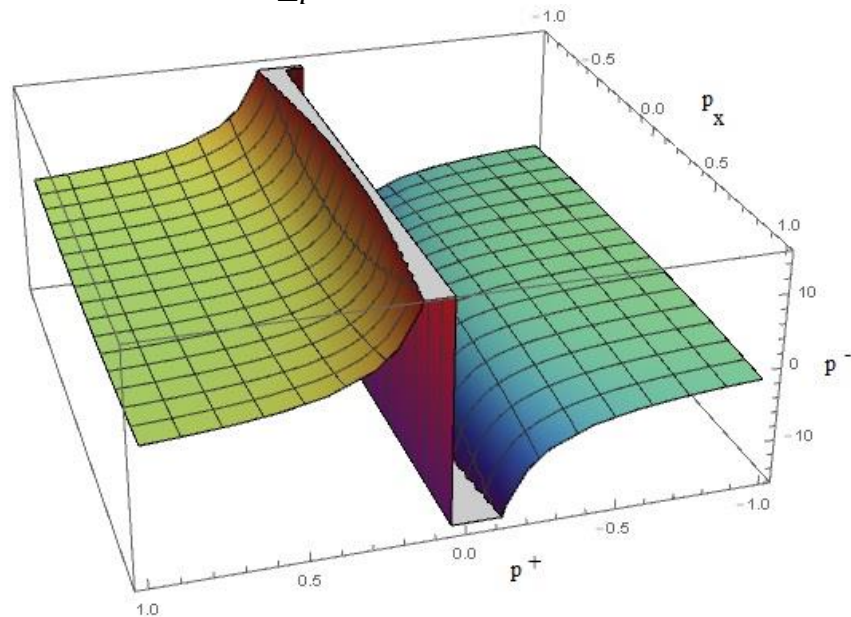


Figure 5 - Zero modes in $p^+ \rightarrow 0$

In Figure 5 we observe that at the limit of $p^+ \rightarrow 0$ the energy associated with the coordinates in the light front diverges, i.e.

$$\lim_{p^+ \rightarrow 0} p^- = \frac{m^2 + \vec{p}_\perp^2}{2p^+} \rightarrow \text{diverge}$$

The system of equations coming from (30) is linear homogeneous and contains trivial solutions $u_1 = 0, u_2 = 0, u_3 = 0$ and $u_4 = 0$. All other solutions, if any, are called non-trivial solutions. For this reason, we select simplicity values for $p^- > 0$ $u_1 = 1, u_2 = 0$ for spin up, and $u_1 = 0, u_2 = 1$ for spin down. In the case $p^- < 0$ one uses $u_3 = 1, u_4 = 0$ for spin up, and $u_3 = 0, u_4 = 1$ for spin down. Thus, we have:

Table 3 - Spinors in a moving referential in the light front.

$p^- > 0$		$p^- < 0$	
Spin up	Spin down	Spin up	Spin down
$u_1 = 1$	$u_1 = 0$	$u_1 = \frac{-p^- - p^+}{p^- - p^+ - \sqrt{2}m}$	$u_1 = \frac{\sqrt{2}(p_x - ip_y)}{p^- - p^+ - \sqrt{2}m}$
$u_2 = 0$	$u_2 = 1$	$u_2 = \frac{\sqrt{2}(p_x + ip_y)}{p^- - p^+ - \sqrt{2}m}$	$u_2 = \frac{p^- + p^+}{p^- - p^+ - \sqrt{2}m}$
$u_3 = \frac{-p^- + p^+}{p^- + p^+ + \sqrt{2}m}$	$u_3 = \frac{\sqrt{2}(p_x - ip_y)}{p^- + p^+ + \sqrt{2}m}$	$u_3 = 1$	$u_3 = 0$
$u_4 = \frac{\sqrt{2}(p_x + ip_y)}{p^- + p^+ + \sqrt{2}m}$	$u_4 = \frac{p^- - p^+}{p^- + p^+ + \sqrt{2}m}$	$u_4 = 0$	$u_4 = 1$

A particle at rest in the Minkowski coordinate system in the space of moments corresponds to the quadrimomentum

$$p^\mu = (p^0, 0, 0, 0) \tag{33}$$

In the light front, the particle at rest in the spaces of the moments $p^\mu|_{FL}$ can be obtained from (12):

$$p^\mu|_{FL} = (p^-, p^+, \vec{p}_\perp) \tag{34}$$

In the inertial system at rest in the coordinates in front of the light, one has the following conditions given:

$$\vec{p} = 0 \Rightarrow \begin{cases} p^1 = 0 \\ p^2 = 0 \\ p^3 = 0 \end{cases} \Rightarrow \vec{p}^\perp = p^1\vec{i} + p^2\vec{j} = 0 \tag{35}$$

The result (35) implies that

$$p^3 = \frac{p^+ - p^-}{\sqrt{2}} = 0 \Rightarrow p^+ = p^- \tag{36}$$

In this way substituting (35), (36) into (12), we obtain

$$p^+ = \frac{m}{\sqrt{2}} \tag{37}$$

Therefore, substituting (30), (31) and (32) in table 3, we obtain the spinors of the particle in a rest frame given by table 4, and it agrees with the results of Table 2.

Table 4 - Spinors in the resting referential in the light front.

$p^- > 0$		$p^- < 0$	
Spin up	Spin down	Spin up	Spin down
$u_1 = 1$	$u_1 = 0$	$u_1 = 0$	$u_1 = 0$
$u_2 = 0$	$u_2 = 1$	$u_2 = 0$	$u_2 = 0$
$u_3 = 0$	$u_3 = 0$	$u_3 = 1$	$u_3 = 0$
$u_4 = 0$	$u_4 = 0$	$u_4 = 0$	$u_4 = 1$

The result of Table 4 shows us that in the particle referential the Minkowski coordinate systems, table 2, and light front are equivalent.

The negative sign for the energy “-“ (negative) in the light front is interpreted as the existence of antiparticles is arbitrary, through the choice of the sign of p^+ . Furthermore, in the coordinates of the light front, since the signs of p^- and p^+ are linked; there is no possibility of the simultaneous appearance of particles and antiparticles. In the light front, particle or antiparticle propagation is defined according to the choice of the sign of p^+ (Sales *et al.*, 2021) and according to causality in the light front (Acevedo *et al.*, 2021).

5. Conclusion

In this paper, an alternative technique for studying the Dirac equation has been presented. This technique is a coordinate shift from Minkowski space-time to a coordinate system known as the light front.

The result shows us, from the propagator point of view, that particles propagating forward in time $x^+ > 0$ on the light front equals the energy of the system for positive to p^- . For propagation into the past $x^+ < 0$ equals negative ($p^+ < 0$).

The limit of this method is that we have a singular point for $p^+ = 0$, which implies

$$\lim_{p^+ \rightarrow 0} p^- \rightarrow \text{diverge}$$

That is the zero modes of the light front theory.

Acknowledgements

To CNPq and FAPESB for their support.

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