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Turbulent Flow Analysis with Banach and Sobolev Spaces in the LES Method Incorporating the Smagorinsky Subgrid-Scale Model

Análise do Escoamento Turbulento com Espaços de Banach e Sobolev no Método LES Incorporando o Modelo de Sub-malha de Smagorinsky

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Resumo
A análise matemática, aplicada neste trabalho, serve de pilar para uma investigação mais ampla sobre a regularidade das Equações de Navier-Stokes. Nesse contexto, esta investigação marca um passo significativo no avanço do modelo de Smagorinsky acoplado à metodologia LES, resultando com base nos Espaços de Banach e de Sobolev, um novo teorema que aponta o caminho para a construção de um modelo de viscosidade anisotrópica (ainda não discutido no presente trabalho). A princípio, o esforço dedicado aqui, visa apresentar uma análise matemática mais abrangente, promovendo uma compreensão mais nivelada do desafio proposto pela regularidade das equações de Navier-Stokes.
Palavras-chave: Modelo Smagorinsky. Espaços de Banach e de Sobolev. Escoamento turbulento.

Abstract
The mathematical analysis, applied in this work, serves as a pillar for a broader investigation on the regularity of the Navier-Stokes Equations. In this context, this investigation marks a significant step forward in the advancement of the Smagorinsky model coupled with the LES methodology, resulting, based on the Banach and Sobolev Spaces, a new theorem that points the way to the construction of an anisotropic viscosity model (not yet discussed in the present work). At first, the effort dedicated here aims to present a more comprehensive mathematical analysis, promoting a more leveled understanding of the challenge posed by the regularity of the Navier-Stokes equations.
Keywords: Smagorinsky model. Banach and Sobolev spaces. Turbulent flow.

1. Introduction

Turbulent formations emerge in both natural occurrences and human endeavors, such as the flow of rivers or the emissions billowing from chimneys. Scrutinizing the dynamics of motion carries importance across domains like aeronautics, meteorology, and engineering. The quantifiable factor referred to as the Reynolds number

$$Re = \frac{UL}{\nu} = \frac{\rho UL}{\mu} \tag{1}$$

(with characteristic velocity U, characteristic length L, kinematic viscosity ν, density ρ and dynamic viscosity μ) is a measure for turbulence of a flow. As demonstred by Reynolds' experiment with piper-flow, a fluid motion featuring a Reynolds number exceeding 4×10^3 displays turbulence, see more in Li Ta-tsien & Yu Wen-ci (1985), Germano (1991), Kolmogorov (1991) and Pope (2000).

2. The Smagorinsky model

To conclude the equations and consequently determine the filtered velocity field $\bar{\mathbf{u}}(\mathbf{x}, t)$ along with the adjusted filtered pressure $\bar{p}(\mathbf{x}, t)$, it is imperative to formulate the anisotropic residual-stress tensor $\tau_{ij}^r(\mathbf{x}, t)$. Among the available models, the Smagorinsky model stands out due to its simplicity and its demonstrated capability to yield satisfactory performance (more details at Pope (2000)).

In the Smagorinsky model, the anisotropic residual-stress tensor $\tau_{ij}^r(\mathbf{x}, t)$ correlates with the filtered strain rate

$$\bar{S}_{ij} = \bar{S}_{ij}(\mathbf{u}) := S_{ij}(\bar{\mathbf{u}}) := 0.5\big(\partial \bar{u}_i + \partial \bar{u}_j\big)\,, \tag{2}$$

as

$$\tau_{ij}^r(\mathbf{x}, t) = -2\nu_r \bar{S}_{ij}\,. \tag{3}$$

This embodies the mathematical representation of the Boussinesq conjecture, which proposes that turbulent fluctuations display dissipative characteristics on an average basis. The mathematical structure shares similarities with that of molecular diffusion, (for further information, see more at Sagaut (2005)). The residual subgrid-scale eddy-viscosity ν_r acts as an artificial viscosity (Sagaut (2005),) and represents the eddy-viscosity of the residual motions. It is modeled as

$$\nu_r = \ell_S^2\big(2\,\bar{S}_{ij}\bar{S}_{ij}\big)^{\frac{1}{2}} = (C_S\Delta)^2\,\big(2\,\bar{S}_{ij}\bar{S}_{ij}\big)^{\frac{1}{2}}. \tag{4}$$

In this context, we encounter the Smagorinsky length scale $\ell_S = C_S\Delta$, the Smagorinsky coefficient C_S, the filter width Δ. Lastly, we can express the filtered momentum equation as follows

$$\partial_t \bar{u}_{ij} + \bar{u}_i \partial_i \bar{u}_j = 2\partial_i\left(\left(\nu + \ell_S^2\big(2\,\bar{S}_{ij}\bar{S}_{ij}\big)^{\frac{1}{2}}\right)\bar{S}_{ij}\right) - \partial_j \bar{p} + \bar{f}_j, j = 1,2,3. \tag{5}$$

3. Mathematical analysis of the Smagorinsky model

In order to conduct a mathematical analysis of the Smagorinsky model, it is essential that the problem is clearly and precisely defined.

3.1 Vector spaces

The Lebesgue space $L^p(\Omega)$, $p \in [1, \infty]$, is the Banach space of measurable functions $\mathbf{v}$ on Ω which satisfy

$$\|\mathbf{v}\|_{L^{m,p}(\Omega)} := \left(\int_\Omega |\mathbf{v}(\mathbf{x})|\,d\mathbf{x}\right)^{\frac{1}{p}} < \infty, if\ p \in [1, \infty), \tag{6}$$

$$\|\mathbf{v}\|_{L^{m,p}(\Omega)} := \ ess\ sup\ |\mathbf{v}(\mathbf{x})| < \infty, if\ p = \infty,$$

For $p = 2$ the Lebesgue space is also a Hilbert space with the scalar product

$$(\mathbf{x}, \mathbf{v}) = \int_{\Omega} \mathbf{v}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) d\mathbf{x}. \tag{7}$$

in the case of one-dimensional functions, the dot signifies straightforward multiplication; however, when dealing with vectors or matrices, it denotes the dot product for vectors or the Frobenius inner product for matrices.

The Sobolev space $W^{m,p}$ is the Banach space of functions for wich

$$\|\mathbf{v}\|_{W^{m,p}(\Omega)} := \left(\sum_{0 \le \lceil \alpha \rceil \le m} \|D^{\alpha}\mathbf{v}\|^{p}_{L^{p}(\Omega)} \right)^{\frac{1}{p}} < \infty, \text{if}\, p \in [1, \infty),$$
$$\|\mathbf{v}\|_{W^{m,p}(\Omega)} := \ max\ \|D^{\alpha}\mathbf{v}\|^{p}_{L^{p}(\Omega)} < \infty, \text{if}\, p = \infty, \tag{8}$$

remains valid, i.e., it can be defined as

$$W^{m,p}(\Omega) = \{\mathbf{v} \in L^{p}(\Omega) : D^{\alpha}\mathbf{v} \in L^{p}(\Omega), \forall |\alpha| \le m\}. \tag{9}$$

Let

$$W^{1,3}{}_{0,div}(\Omega) = \{\mathbf{v} \in W^{1,3}(\Omega) : \mathbf{v}|_{\Gamma} = 0, \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega\}, \tag{10}$$

be the divergence-free Sobolev space where functions vanish on the boundary $\Gamma = \partial\Omega$,

$$H^{1}\big(0, \mathrm{T}; L^{2}(\Omega)\big) := W^{1,2}\big(0, \mathrm{T}; L^{2}(\Omega)\big) \tag{11}$$

a Sobolev space that is also a Hilbert space and

$$V \coloneqq H^{1}\big(0, \mathrm{T}; L^{2}(\Omega)\big) \cap L^{3}\left(0, T; W^{1,3}{}_{0,div}(\Omega)\right), \tag{12}$$

a Banach space with the norm

$$\|\mathbf{v}\|_{V} = \|\nabla\mathbf{v}\|_{L^{3}\left(0,T;L^{3}(\Omega)\right)} + \|\partial_{t}\mathbf{v}\|_{L^{2}\left(0,T;L^{2}(\Omega)\right)}. \tag{13}$$

3.2 Strong and weak formulation of Navier-Stokes Equation

Consider the Navier-Stoke Equation with the conditions

$$\partial_{t}\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu\nabla \cdot \nabla\mathbf{u} - \frac{1}{\rho}\nabla \mathrm{P} + \mathrm{f}, \text{in } \Omega \times (0, T],$$
$$\nabla \cdot \mathbf{u} = 0, \text{ in } \Omega \times [0, T], \tag{14}$$

$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})$, in Ω
$\mathbf{u} = 0$ on $\Gamma \times [0,T]$,
$\int_{\Omega} P \,\mathrm{d}\mathbf{x} = 0$, in $\Omega \times (0,T]$,

with $\Gamma = \partial\Omega$. The first and second equations correspond to the momentum equation and continuity equation from above. The initial flow field $\mathbf{u}_0(\mathbf{x})$ is also divergence-free, i.e., $\nabla \cdot \mathbf{u}_0 = 0$ in Ω. The fourth equation is the no slip boundary condition. It relies on the supposition that the fluid does not permeate or slide along the wall. Without the last equation, the pressure P would only be determined up to a constant, according Sagaut (2005), Hunt & Vassilicos (1991).

Filtering Eqs. (14) and using a similar condition for the modified filtered pressure, we get

$$
\begin{aligned}
&\partial_t\bar{\mathbf{u}} + (\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}} = \nabla(\nu+\nu_r)\nabla\bar{\mathbf{u}} - \nabla\bar{p} + \bar{\boldsymbol{f}}, \text{in } \Omega\times(0,T],\\
&\nabla\cdot\bar{\mathbf{u}} = 0, \text{ in } \Omega\times[0,T],\\
&\bar{\mathbf{u}}(\mathbf{x},0) = \bar{\mathbf{u}}_0(\mathbf{x}), \text{ in } \Omega\\
&\bar{\mathbf{u}} = 0 \text{ on } \Gamma\times[0,T],\\
&\int_{\Omega} \bar{p}\,\mathrm{d}\mathbf{x} = 0, \text{ in } \Omega\times(0,T],
\end{aligned}
\tag{15}
$$

by multiplying the first equation with $\boldsymbol{v} \in V$ and integrating over time and space, we achieve a weak formulation. Now, let $\bar{\boldsymbol{f}} \in L^2\big(0,\mathrm{T};L^2(\Omega)\big)$. Find $\bar{\mathbf{u}} \in V$ that satisfies $\bar{\mathbf{u}}(\mathbf{x},0) = \bar{\mathbf{u}}_0(\mathbf{x}) \in W^{1,3}{}_{0,div}(\Omega)$ and

$$
\int_0^T (\partial_t\bar{\mathbf{u}} + (\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}},\mathbf{v}) + \big((\nu+\nu_r)\nabla\bar{\mathbf{u}},\nabla\mathbf{v}\big)dt = \int_0^T \big(\bar{\boldsymbol{f}},\mathbf{v}\big)dt\,, \tag{16}
$$

for all $\mathbf{v} \in V$, with $(\cdot,\cdot)$ denoting the $L^2(\Omega)$ scalar product.

3.2 Asymptotic behavior and regularity

Let us first introduce some standard notations and function spaces which will be used in the following analysis. We denote

$$\mathcal{V} = \{\varphi \in \mathcal{D}(\Omega)^3, \nabla\cdot\,\varphi = 0\},$$

$$H = \text{the closure of } \mathcal{V} \text{ in } L^2(\Omega)^3,$$

$$V = \text{the closurr of } \mathcal{V} \text{ in } W^{1,3}(\Omega)^3,$$

where $L^2(\Omega)^2$ is the space of functions which are square integrable over Ω with respect to the Lebesgue measure and $W^{1,3}(\Omega)^3$ is the L^3 Sobolev space. H is a Hilbert space with respect to the inner product. We will use the notation V' for the dual space of V, $\|\cdot\|_V$, for the induced norm and $\langle\cdot,\cdot\rangle$ for the duality product. For spaces of functions which depend on both time and space variables, we will frequently use the two following spaces: (**i**) $C([0,T];X)$ space of continuous functions $u\colon [0,T] \longrightarrow X$, where X is a Banach space with the norm denoted by $|\cdot|_X$. (**ii**) $L^p(0,\mathrm{T};X)$ the space of strongly measurable functions $u\colon]0,T[\longrightarrow X$ with a finite norm

$$|u|^p_{L^p(0,T;X)} := \int_0^T |u|^p_X\,dt < \infty.$$

In the case $p = \infty$, the norm is defined by

$$|u|^{p}_{L^{\infty}(0,T;X)} \coloneqq ess\ \sup_{t\in]0,T[}|u(t)|_X.$$

Finally, we will denote by $|\cdot|_p$ the usual norm in $L^p(\Omega)$.

Theorem. Let $\mathbf{u}_0 \in H$ and $\boldsymbol{f} \in L^{\frac{3}{2}}(0,T;V')$. Then for any $\mathcal{T} > 0$, the problem $(\mathcal{S})$ has a unique weak solution on $[0,T]$. Moreover, if $\mathbf{u}_0 \in V$ then the unique weak solution is in $L^{\infty}(0,\mathrm{T};W^{1,3}(\Omega)^3)$.

Proof. To prove the existence of a weak solution we used a classical Galerkin method. We omit it, since it is straightforward from the proof done in Lions (2008) and Jiroveanu (2002). We only present here, the *proof of uniqueness*. Let us suppose that there exist two weak solutions $\mathbf{u}$ and $\mathbf{v}$ to problem $(\mathcal{S})$, with the same initial condition $\mathbf{u}_0 \in H$ and let $\mathbf{w} = \mathbf{u} - \mathbf{v}$. After subtracting the weak formulation for $\mathbf{v}$ from the one for $\mathbf{u}$ and talking $\mathbf{w}$ as test functions in the resulting equation, we get:

$$\frac{1}{2}\frac{d}{dt}\mathbf{w}_2^2 + \sum_{i,j=1}^{3}\int_{\Omega}\left[\mathcal{T}_{ij}\big(S(\mathbf{u})\big) - \mathcal{T}_{ij}\big(S(\mathbf{v})\big)\right]S_{ij}(\mathbf{w})d\mathbf{x} = -\int_{\Omega}(\mathbf{w}\cdot\nabla)\mathbf{u}\,\mathbf{w}d\mathbf{x}. \tag{17}$$

Moreover, from the definition of the tensor $\mathcal{T}$ (see more in Pope (2000), Hoffman & Johnson (2006)), we have:

$$\sum_{i,j=1}^{3}\int_{\Omega}\left[\mathcal{T}_{ij}\big(S(\mathbf{u})\big) - \mathcal{T}_{ij}\big(S(\mathbf{v})\big)\right]d\mathbf{x} = c_1\sum_{i,j}^{3}\int_{\Omega}\left|S_{ij}(\mathbf{w})\right|^2 d\mathbf{x}\,, \tag{18}$$

with $c_1 > 0$.

Using Korn's inequality

$$\left(\int_{\Omega}|S(\mathbf{u})|^p\,d\mathbf{x}\right)^{\frac{1}{p}} \geq C_p|\nabla\mathbf{u}|_p$$

for $\mathbf{u} \in W_0^{1,p}$ with $C_p > 0$ $(1 < p < \infty)$ and Hölder's inequality we obtain from Eq. (18)

$$\frac{1}{2}\frac{d}{dt}|\mathbf{w}|_2^2 + c_2|\nabla\mathbf{w}|_2^2 \leq \int_{\Omega}|\mathbf{w}|^2\,|\nabla\mathbf{u}|d\mathbf{x} \leq |\nabla\mathbf{u}|_3|\mathbf{w}|_3^2\,. \tag{19}$$

In three dimensions we have the embedding

$$H^1(\Omega) \subset L^6(\Omega)$$

from which we deduce

$$|\boldsymbol{w}|_3 \leq |\boldsymbol{w}|_2^{\frac{1}{2}} |\boldsymbol{w}|_6^{\frac{1}{2}} \leq c_3 |\boldsymbol{w}|_2^{\frac{1}{2}} |\nabla \boldsymbol{w}|_2^{\frac{1}{2}}.$$

Moreover, it follows from Eq. (19), via Young's inequality, that

$$\frac{d}{dt}|\mathbf{w}|_2^2 + c_4|\nabla \mathbf{w}|_2^2 \leq c_5|\nabla \mathbf{u}|_3^2 \leq |\mathbf{w}|_3^2. \tag{20}$$

Since the functions $g(t) = |\nabla \mathbf{u}|_3^2$ is integrable on $]0, T[$ and $\boldsymbol{w}(0) = 0$, using Gronwall's inequality we get

$$|\boldsymbol{w}(t)|_2^2$$

on $[0, T]$ and thus uniqueness of the solution to problem $(\mathcal{S})$. □

The uniform in time regularity is related to the asymptotic behavior of the solution that we now consider.

Let $\mathbf{u}_0 \in H$ and suppose now that $\boldsymbol{f} \in L^2(\Omega)^3$ is time independent. According *Theorem*, the unique weak solution is continous

$$\mathbf{u} \in C\big((0, T); H\big).$$

Consequently, we can define the family of operators $\big(S(t)\big)_{t\geq 0}$ by

$$\begin{aligned} S(t)\colon & H \longrightarrow H \\ & \mathbf{u}_0 \mapsto S(t)_{\mathbf{u}_0 = \mathbf{u}(t)} \end{aligned} \tag{21}$$

is the solution to problem $(\mathcal{S})$.

4. Conclusion

In conclusion, this study has undertaken a rigorous reexamination of the Smagorinsky model, shedding light on the sub-grid's mathematical formulation through asymptotic analysis of the LES model. The elucidation of this mathematical analysis not only serves as a foundational element but also paves the way for a more extensive investigation into regularity of the Navier-Stokes Equations.

It is our firm belief that this investigation marks a significant step towards advancing the Smagorinsky model, with the anticipation that future research will yield an anisotropic viscosity model for turbulent flow, ultimately addressing the elusive question of regularity within the Navier-Stokes Equations. This dedicated effort aims to present a comprehensive mathematical analysis, inspiring further exploration and fostering a deeper understanding of the enduring challenge posed by the regularity of the Navier-Stokes equations.

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