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Treatment for regularity of the Navier-Stokes equations based on Banach and Sobolev functional spaces coupled to anisotropic viscosity for analysis of vorticity transport

Tratamento da regularidade das equações de Navier-Stokes baseado nos espaços funcionais de Banach e Sobolev acoplados à viscosidade anisotrópica para análise do transporte de vorticidade

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Resumo
A análise matemática empregada neste estudo constitui um fundamento essencial para uma investigação mais ampla sobre a regularidade das Equações de Navier-Stokes. Dentro desse contexto, este trabalho representa um avanço significativo junto ao modelo de Smagorinsky integrado à metodologia LES. Utilizando os espaços funcionais de Banach e Sobolev, desenvolvemos um novo teorema que aponta uma trajetória para a criação de um modelo de viscosidade anisotrópica, formulado no presente trabalho. Inicialmente, nosso esforço se concentra em fornecer uma análise matemática abrangente, com o objetivo de promover uma compreensão mais profunda do desafio inerente à regularidade das equações de Navier-Stokes.
Palavras-chave: Modelo Smagorinsky. Espaços funcionais. Viscosidade anisotrópica.

Abstract
The mathematical analysis employed in this study constitutes an essential foundation for a broader investigation into the regularity of the Navier-Stokes Equations. Within this context, this work represents a significant advance with the Smagorinsky model integrated into the LES methodology. Using the Banach and Sobolev functional spaces, we developed a new theorem that points out a path towards the creation of an anisotropic viscosity model, formulated in the present work. Initially, our effort focuses on providing a comprehensive mathematical analysis, with the aim of promoting a deeper understanding of the challenge inherent in the regularity of the Navier-Stokes equations.
Keywords: Smagorinsky model. Functional spaces. Turbulent flow. Anisotropic viscosity.

1. Introduction

Turbulent formations emerge in both natural occurrences and human endeavors, such as the flow of rivers or the emissions billowing from chimneys. Scrutinizing the dynamics of motion carries importance across domains like aeronautics, meteorology, and engineering. The quantifiable factor referred to as the Reynolds number

$$Re = \frac{UL}{\nu} = \frac{\rho UL}{\mu} \tag{1}$$

(with characteristic velocity U, characteristic length L, kinematic viscosity ν, density ρ and dynamic viscosity μ) is a measure for turbulence of a flow. As demonstred by Reynolds' experiment with piper-flow, a fluid motion featuring a Reynolds number exceeding 4×10^3 displays turbulence, see more in Li Ta-tsien & Yu Wen-ci (1985), Germano (1991), Kolmogorov (1991) and Pope (2000).

2. The Smagorinsky model

According to the work of Santos, R.D.C. dos, & Sales, J.H.O. (2023), to conclude the equations and consequently determine the filtered velocity field $\bar{\mathbf{u}}(\mathbf{x}, t)$ along with the adjusted filtered pressure $\bar{p}(\mathbf{x}, t)$, it is imperative to formulate the anisotropic residual-stress tensor $\tau_{ij}^{r}(\mathbf{x}, t)$. Among the available models, the Smagorinsky model stands out due to its simplicity and its demonstrated capability to yield satisfactory performance (more details at Pope (2000)).

In the Smagorinsky model, the anisotropic residual-stress tensor $\tau_{ij}^{r}(\mathbf{x}, t)$ correlates with the filtered strain rate

$$\bar{S}_{ij} = \bar{S}_{ij}(\mathbf{u}) := S_{ij}(\bar{\mathbf{u}}) := 0.5\left(\partial \bar{u}_i + \partial \bar{u}_j\right), \tag{2}$$

as

$$\tau_{ij}^{r}(\mathbf{x}, t) = -2\nu_r \bar{S}_{ij}\,. \tag{3}$$

This embodies the mathematical representation of the Boussinesq conjecture, which proposes that turbulent fluctuations display dissipative characteristics on an average basis. The mathematical structure shares similarities with that of molecular diffusion, (for further information, see more at Sagaut (2005)). The residual subgrid-scale eddy-viscosity ν_r acts as an artificial viscosity (Sagaut (2005),) and represents the eddy-viscosity of the residual motions. It is modeled as

$$\nu_r = \ell_S^2\left(2\,\bar{S}_{ij}\bar{S}_{ij}\right)^{\frac{1}{2}} = (C_S\Delta)^2\left(2\,\bar{S}_{ij}\bar{S}_{ij}\right)^{\frac{1}{2}}. \tag{4}$$

In this context, we encounter the Smagorinsky length scale $\ell_S = C_S\Delta$, the Smagorinsky coefficient C_S, the filter width Δ. Lastly, we can express the filtered momentum equation as follows

$$\partial_t \bar{u}_{ij} + \bar{u}_i \partial_i \bar{u}_j = 2\partial_i\left(\left(\nu + \ell_S^2\left(2\,\bar{S}_{ij}\bar{S}_{ij}\right)^{\frac{1}{2}}\right)\bar{S}_{ij}\right) - \partial_j \bar{p} + \bar{f}_j, j = 1,2,3. \tag{5}$$

3. Mathematical analysis of the Smagorinsky model

In order to conduct a mathematical analysis of the Smagorinsky model, it is essential that the problem is clearly and precisely defined.

3.1 Vector spaces

The Lebesgue space $L^p(\Omega)$, $p \in [1, \infty]$, is the Banach space of measurable functions $\mathbf{v}$ on Ω which satisfy

$$\|\mathbf{v}\|_{L^{m,p}(\Omega)} := \left(\int_{\Omega} |\mathbf{v}(\mathbf{x})| d\mathbf{x} \right)^{\frac{1}{p}} < \infty, if\ p \in [1, \infty), \tag{6}$$

$$\|\mathbf{v}\|_{L^{m,p}(\Omega)} := \ ess\ sup\ |\mathbf{v}(\mathbf{x})| < \infty, if\ p = \infty,$$

For $p = 2$ the Lebesgue space is also a Hilbert space with the scalar product

$$(\mathbf{x}, \mathbf{v}) = \int_{\Omega} \mathbf{v}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) d\mathbf{x}. \tag{7}$$

in the case of one-dimensional functions, the dot signifies straightforward multiplication; however, when dealing with vectors or matrices, it denotes the dot product for vectors or the Frobenius inner product for matrices.

The Sobolev space $W^{m,p}$ is the Banach space of functions for which

$$\|\mathbf{v}\|_{W^{m,p}(\Omega)} := \left(\sum_{0 \le \lceil \alpha \rceil \le m} \|D^{\alpha}\mathbf{v}\|_{L^p(\Omega)}^{p} \right)^{\frac{1}{p}} < \infty, \text{if}\ p \in [1, \infty), \tag{8}$$

$$\|\mathbf{v}\|_{W^{m,p}(\Omega)} := \ max\ \|D^{\alpha}\mathbf{v}\|_{L^p(\Omega)}^{p} < \infty, \text{if}\ p = \infty,$$

remains valid, i.e., it can be defined as

$$W^{m,p}(\Omega) = \{\mathbf{v} \in L^p(\Omega) : D^{\alpha}\mathbf{v} \in L^p(\Omega), \forall |\alpha| \le m\}. \tag{9}$$

Let

$$W^{1,3}{}_{0,div}(\Omega) = \{\mathbf{v} \in W^{1,3}(\Omega) : \mathbf{v}|_{\Gamma} = 0, \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega\}, \tag{10}$$

be the divergence-free Sobolev space where functions vanish on the boundary $\Gamma = \partial\Omega$,

$$H^1\big(0, \mathrm{T}; L^2(\Omega)\big) := W^{1,2}\big(0, \mathrm{T}; L^2(\Omega)\big) \tag{11}$$

a Sobolev space that is also a Hilbert space and

$$V \coloneqq H^1\big(0, \mathrm{T}; L^2(\Omega)\big) \cap L^3\left(0, T; W^{1,3}{}_{0,div}(\Omega)\right), \tag{12}$$

a Banach space with the norm

$$\|\mathbf{v}\|_V = \|\nabla\mathbf{v}\|_{L^3\left(0,T;L^3(\Omega)\right)} + \|\partial_t\mathbf{v}\|_{L^2\left(0,T;L^2(\Omega)\right)}. \tag{13}$$

3.2 Strong and weak formulation of Navier-Stokes Equation

Consider the Navier-Stoke Equation with the conditions

$$\begin{aligned} &\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\nabla\cdot\nabla\mathbf{u} - \frac{1}{\rho}\nabla \mathrm{P} + \mathrm{f}, \text{in } \Omega\times(0,T],\\ &\nabla\cdot\mathbf{u} = 0, \text{ in } \Omega\times[0,T],\\ &\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \text{ in } \Omega\\ &\mathbf{u} = 0 \text{ on } \Gamma\times[0,T],\\ &\textstyle\int_{\boldsymbol{\Omega}} P\,\mathrm{d}\mathbf{x} = 0, \text{ in } \Omega\times(0,T], \end{aligned} \tag{14}$$

with $\Gamma = \partial\Omega$. The first and second equations correspond to the momentum equation and continuity equation from above. The initial flow field $\mathbf{u}_0(\mathbf{x})$ is also divergence-free, i.e., $\nabla\cdot\mathbf{u}_0 = 0$ in Ω. The fourth equation is the no slip boundary condition. It relies on the supposition that the fluid does not permeate or slide along the wall. Without the last equation, the pressure P would only be determined up to a constant, according Sagaut (2005), Hunt & Vassilicos (1991).

Filtering Eqs. (14) and using a similar condition for the modified filtered pressure, we get

$$\begin{aligned} &\partial_t\bar{\mathbf{u}} + (\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}} = \nabla(\nu+\nu_r)\nabla\bar{\mathbf{u}} - \nabla\bar{p} + \bar{\boldsymbol{f}}, \text{in } \Omega\times(0,T],\\ &\nabla\cdot\bar{\mathbf{u}} = 0, \text{ in } \Omega\times[0,T],\\ &\bar{\mathbf{u}}(\mathbf{x},0) = \bar{\mathbf{u}}_0(\mathbf{x}), \text{ in } \Omega\\ &\bar{\mathbf{u}} = 0 \text{ on } \Gamma\times[0,T],\\ &\textstyle\int_{\boldsymbol{\Omega}} \bar{p}\,\mathrm{d}\mathbf{x} = 0, \text{ in } \Omega\times(0,T], \end{aligned} \tag{15}$$

by multiplying the first equation with $\boldsymbol{v}\in V$ and integrating over time and space, we achieve a weak formulation. Now, let $\bar{\boldsymbol{f}} \in L^2\big(0,\mathrm{T}; L^2(\Omega)\big)$. Find $\bar{\mathbf{u}}\in V$ that satisfies $\bar{\mathbf{u}}(\mathbf{x},0) = \bar{\mathbf{u}}_0(\mathbf{x}) \in W^{1,3}{}_{0,div}(\Omega)$ and

$$\int_0^T (\partial_t\bar{\mathbf{u}} + (\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}},\mathbf{v}) + \big((\nu+\nu_r)\nabla\bar{\mathbf{u}},\nabla\mathbf{v}\big)dt = \int_0^T \big(\bar{\boldsymbol{f}},\mathbf{v}\big)dt\,, \tag{16}$$

for all $\mathbf{v}\in V$, with $(\cdot,\cdot)$ denoting the $L^2(\Omega)$ scalar product.

3.2 Asymptotic behavior and regularity

Let us first introduce some standard notations and function spaces which will be used in the following analysis. We denote $\mathcal{V} = \{\varphi\in\mathcal{D}(\Omega)^3, \nabla\cdot\ \varphi = 0\}$, $H =$ the closure of $\mathcal{V}$ in $L^2(\Omega)^3$, $V =$ the closurr of $\mathcal{V}$ in $W^{1,3}(\Omega)^3$, where $L^2(\Omega)^2$ is the space of functions which are square integrable over Ω with respect to the Lebesgue measure and $W^{1,3}(\Omega)^3$ is the L^3 Sobolev space. H is a Hilbert space with respect to the inner product. We will use the notation V' for the dual space of V, $\|\cdot\|_V$, for the induced norm and $\langle\cdot,\cdot\rangle$ for the duality product. For spaces of functions which depend on both time and space variables, we will frequently use the two following spaces: **(i)** $C([0,T];X)$ space of continuous functions $u: [0,T] \longrightarrow X$, where X is a Banach space with the norm denoted by $|\cdot|_X$. **(ii)** $L^p(0,\mathrm{T};X)$ the space of strongly measurable functions $u:]0,T[\longrightarrow X$ with a finite norm

$$|u|^p_{L^p(0,T;X)} := \int_0^T |u|^p_X \, dt < \infty.$$

In the case $p = \infty$, the norm is defined by

$$|u|^p_{L^\infty(0,T;X)} := ess \, \sup_{t \in]0,T[} |u(t)|_X.$$

Finally, we will denote by $|\cdot|_p$ the usual norm in $L^p(\Omega)$.

Theorem. Let $\mathbf{u}_0 \in H$ and $\boldsymbol{f} \in L^{\frac{3}{2}}(0,T;V')$. Then for any $\mathcal{T} > 0$, the problem $(\mathcal{S})$ has a unique weak solution on $[0,T]$. Moreover, if $\mathbf{u}_0 \in V$ then the unique weak solution is in $L^\infty(0,\mathrm{T};W^{1,3}(\Omega)^3)$.

Proof. To prove the existence of a weak solution we used a classical Galerkin method. We omit it, since it is straightforward from the proof done in Lions (2008) and Jiroveanu (2002). We only present here, the *proof of uniqueness*. Let us suppose that there exist two weak solutions $\mathbf{u}$ and $\mathbf{v}$ to problem $(\mathcal{S})$, with the same initial condition $\mathbf{u}_0 \in H$ and let $\mathbf{w} = \mathbf{u} - \mathbf{v}$. After subtracting the weak formulation for $\mathbf{v}$ from the one for $\mathbf{u}$ and talking $\mathbf{w}$ as test functions in the resulting equation, we get:

$$\frac{1}{2}\frac{d}{dt}\mathbf{w}_2^2 + \sum_{i,j=1}^{3} \int_\Omega \left[\mathcal{T}_{ij}\big(S(\mathbf{u})\big) - \mathcal{T}_{ij}\big(S(\mathbf{v})\big)\right] S_{ij}(\mathbf{w}) d\mathbf{x} = -\int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{u}\,\mathbf{w} d\mathbf{x}. \tag{17}$$

Moreover, from the definition of the tensor $\mathcal{T}$ (see more in Pope (2000), Hoffman & Johnson (2006) and Santos, R.D.C. dos, & Sales, J.H.O. (2023)), we have:

$$\sum_{i,j=1}^{3} \int_\Omega \left[\mathcal{T}_{ij}\big(S(\mathbf{u})\big) - \mathcal{T}_{ij}\big(S(\mathbf{v})\big)\right] d\mathbf{x} = c_1 \sum_{i,j}^{3} \int_\Omega \left|S_{ij}(\mathbf{w})\right|^2 d\mathbf{x}\,, \tag{18}$$

with $c_1 > 0$.

Using Korn's inequality

$$\left(\int_\Omega |S(\mathbf{u})|^p \, d\mathbf{x}\right)^{\frac{1}{p}} \geq C_p |\nabla \mathbf{u}|_p$$

for $\mathbf{u} \in W_0^{1,p}$ with $C_p > 0$ $(1 < p < \infty)$ and Hölder's inequality we obtain from Eq. (18)

$$\frac{1}{2}\frac{d}{dt}|\mathbf{w}|_2^2 + c_2 |\nabla \mathbf{w}|_2^2 \leq \int_\Omega |\mathbf{w}|^2 \, |\nabla \mathbf{u}| d\mathbf{x} \leq |\nabla \mathbf{u}|_3 |\mathbf{w}|_3^2\,. \tag{19}$$

In three dimensions we have the embedding

$$H^1(\Omega) \subset L^6(\Omega)$$

from which we deduce

$$|\boldsymbol{w}|_3 \leq |\boldsymbol{w}|_2^{\frac{1}{2}} |\boldsymbol{w}|_6^{\frac{1}{2}} \leq c_3 |\boldsymbol{w}|_2^{\frac{1}{2}} |\nabla \boldsymbol{w}|_2^{\frac{1}{2}}.$$

Moreover, it follows from Eq. (19), via Young's inequality, that

$$\frac{d}{dt} |\mathbf{w}|_2^2 + c_4 |\nabla \mathbf{w}|_2^2 \leq c_5 |\nabla \mathbf{u}|_3^2 \leq |\mathbf{w}|_3^2. \tag{20}$$

Since the functions $g(t) = |\nabla \mathbf{u}|_3^2$ is integrable on $]0, T[$ and $\boldsymbol{w}(0) = 0$, using Gronwall's inequality we get

$$|\boldsymbol{w}(t)|_2^2$$

on $[0, T]$ and thus uniqueness of the solution to problem $(\mathcal{S})$.

The uniform in time regularity is related to the asymptotic behavior of the solution that we now consider.

Let $\mathbf{u}_0 \in H$ and suppose now that $\boldsymbol{f} \in L^2(\Omega)^3$ is time independent. According *Theorem*, the unique weak solution is continuous

$$\mathbf{u} \in C\big((0, T); H\big).$$

Consequently, we can define the family of operators $\big(S(t)\big)_{t \geq 0}$ by

$$\begin{aligned} S(t)&: H \longrightarrow H \\ &\mathbf{u}_0 \mapsto S(t)_{\mathbf{u}_0 = \mathbf{u}(t)} \end{aligned} \tag{21}$$

is the solution to problem $(\mathcal{S})$.

4. Smagorinsky refined model

To enhance comprehension of turbulence mechanisms, it is advantageous to examine the velocity-vorticity representation of the Navier–Stokes equations, as it accentuates the distinctions between two and three-dimensional scenarios. Computing the curl of the Navier–Stokes equations results in

$$\partial_t \omega + (\mathbf{u} \cdot \nabla)\omega = (\omega \cdot \nabla)\mathbf{u} + \nu \Delta \omega, \tag{22}$$

where $\omega = \nabla \times \mathbf{u}$ is the vorticity. This delineates the generation and conveyance of vorticity within a generic three-dimensional flow. The final term on the right-hand side elucidates the impacts of viscous diffusion on the vorticity dispersion. In three-dimensional flows with high Reynolds numbers, the viscous dispersion of vorticity is primarily governed by vortex elongation, represented in Eq. (22) by the term $(\omega \cdot \nabla)\mathbf{u}$. This is posited as the predominant mechanism in turbulence dynamics. Through this mechanism, turbulent energy is cascaded from larger to smaller scales.

The stretching component is accountable for both the amplification and realignment of vorticity, potentially playing a role in the generation of finite-time singularities. In two-dimensional flow, vorticity essentially assumes the role of a passive scalar, tracking fluid particle trajectories (in turbulent flow, the influence of viscosity is minimal). In such instances, the vorticity vector is confined to a plane perpendicular to the flow, and its magnitude remains consistently bounded.

If ω is divergenceless, it is feasible to reconstruct the velocity from the vorticity field using the stream function Ψ that satisfies $-\nabla^2\Psi = \omega$, by taking $\mathbf{u} = \nabla \times \Psi$. Hence, one derives the subsequent equation for the velocity field, recognized in literature as the Biot-Savart law (see more in Chorin (2013)),

$$\mathbf{u}(\boldsymbol{x}) = -\frac{1}{4\pi}\int_{\Omega} \frac{(\boldsymbol{x}-\boldsymbol{x}') \times \omega(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3} d\boldsymbol{x}'. \tag{23}$$

A comparable method to grid based LES in primitive variables can be achieved for the velocity-vorticity formulation. When filtering the vorticity transport equations (22), we obtain:

$$\partial_t\overline{\omega}_i + (\bar{u}_j \cdot \partial_j)\overline{\omega}_i = (\overline{\omega}_i \cdot \partial_j)\bar{u}_i + \nu\nabla^2\overline{\omega}_i - \partial_j T_{ij}, \tag{24}$$

where $T_{ij} = (\overline{\omega_i u_j} - \overline{\omega}_i\overline{u}_i) - (\overline{u_i\omega_j} - \overline{u}_i\overline{\omega}_j)$ is the subgrid-scale vorticity stress, which accommodates the influence of unresolved fluctuations in velocity and vorticity. As for the filtered Navier-Stokes equations in primitive variables, it becomes necessary to furnish a model for the vorticity stress T in order to complete the filtered vorticity transport equation (24). A suitable closure model for this situation is the vorticity adaptation of the Smagorinsky model, given by $T = \nu_t\nabla\omega$ with $\nu_t = (C_s\Delta)^2|\omega|$. As previously noted, it has been highlighted that the Smagorinsky model exhibits excessive dissipation.

To grasp the connections between the gradient model and the issue of regularity in the Navier-Stokes equations, let's delve into the enstrophy budget stemming from the Navier-Stokes equations. Multiplying the Eq. (22) by ω, because $\mathbf{u}$ is divergence-free, one obtains

$$\frac{d}{dt}|\omega|^2 = \int_{\Omega} (\omega \cdot \nabla\boldsymbol{u}) \cdot \omega dx - \nu \int_{\Omega} |\nabla\omega|^2 dx\,. \tag{25}$$

The estimation of the stretching term's contribution can be conducted in the following manner. If we denote by $\boldsymbol{S}^+$ the positive part (symmetric) of the tensor $\boldsymbol{S}$ (we recall that $S_{ij} = 1/2\,(\partial_i u_j + \partial_j u_i)$), we obtain

$$\int_{\Omega} (\omega \cdot \nabla\boldsymbol{u}) \cdot \omega dx = \int_{\Omega} \omega_i S_{ij}\omega_j dx \le \int_{\Omega} \omega_i S_{ij}^{+}\omega_j dx\,. \tag{25}$$

Consequently, enstrophy could experience growth, potentially leading to vorticity amplification, when the vorticity aligns with directions corresponding to positive eigenvalues of $\boldsymbol{S}$. If there is a desire to restrict the growth of enstrophy to preserve smooth solutions, this inequality indicates the need for an eddy viscosity tensor with a magnitude proportional to $\boldsymbol{S}^{+}$.

In Vreman (1995), we initiate here with the gradient model, given by

$$\tau_{ij} = (C_a\Delta)^2 \partial_k u_i \partial_k u_j, \tag{26}$$

we first write

$$\Delta^2 \partial_k u_i(\boldsymbol{x}) \partial_k u_j(\boldsymbol{x}) = \Delta^{-3} \int \partial_k u_i(\boldsymbol{x}) \partial_l u_j(\boldsymbol{x})(y_k - x_k)(y_l - x_l)\zeta\left(\frac{\boldsymbol{y}-\boldsymbol{x}}{\Delta}\right) dy\,, \tag{27}$$

where we recall that the summation of repeated indices is implied. Taylor expansions of u_i and u_j, around $\boldsymbol{x}$, yield

$$\Delta^2 \partial_k u_i(\boldsymbol{x}) \partial_k u_j(\boldsymbol{x}) = \Delta^{-3}\left\{\int \left[u_j(\boldsymbol{y}) - u_j(\boldsymbol{x})\right]\left[u_i(\boldsymbol{y}) - u_i(\boldsymbol{x})\right]\zeta\left(\frac{\boldsymbol{y}-\boldsymbol{x}}{\Delta}\right) dy + O(\Delta^2)\right\}. \tag{28}$$

We subsequently compute the divergence of Eq. (28) to obtain

$$\partial_j\left[\partial_k u_i(x)\partial_k u_j(x)\right] \simeq A_i + B_i,$$

following the removal of the component associated with the divergence of **u,** where

$$A_i = -\Delta^{-3}\int \left[u_j(\boldsymbol{y}) - u_j(\boldsymbol{x})\right]\partial_j u_i(x)\zeta\left(\frac{\boldsymbol{y}-\boldsymbol{x}}{\Delta}\right) dy\,,$$

$$B_i = -\Delta^{-4}\int \left[u_j(\boldsymbol{y}) - u_j(\boldsymbol{x})\right]\left[u_i(\boldsymbol{y}) - u_i(\boldsymbol{x})\right]\partial_j\zeta\left(\frac{\boldsymbol{y}-\boldsymbol{x}}{\Delta}\right) dy\,.$$

It is easily seen that A_i is a convective term: if one sets

$$\lambda = \int \zeta(\boldsymbol{y})d\boldsymbol{y}\,,$$

and

$$\hat{u}(x) = \frac{1}{\lambda\Delta^3}\int \boldsymbol{u}(\boldsymbol{y})\zeta\left(\frac{\boldsymbol{y}-\boldsymbol{x}}{\Delta}\right) dy\,,$$

then A_i can be rewritten as $(\hat{\boldsymbol{u}} - \boldsymbol{u})\nabla u_i$. Therefore, it has no impact on the energy equilibrium. Because we aim to depict the energy transfer between different scales, our focus will solely be on B_i. We arrive at the subsequent representation of the gradient model

$$\partial_j \tau_{ij} \simeq C_a^2\Delta^{-4}\int \left[u_j(\boldsymbol{y}) - u_j(\boldsymbol{x})\right]\left[u_i(\boldsymbol{y}) - u_i(\boldsymbol{x})\right]\zeta\left(\frac{\boldsymbol{y}-\boldsymbol{x}}{\Delta}\right) dy. \tag{29}$$

To determine the beneficial impact of this model on energy dissipation, we calculate the product of this quantity by $\mathbf{u}$ and integrate over Ω, and using the symmetry of ζ, we are left with

$$\frac{1}{2}C_a^2\Delta^{-4}\int\left[u_j(\boldsymbol{y})-u_j(\boldsymbol{x})\right]\cdot\nabla\zeta\left(\frac{\boldsymbol{y}-\boldsymbol{x}}{\Delta}\right)|\boldsymbol{u}(\boldsymbol{y})-\boldsymbol{u}(\boldsymbol{x})|^2 d\boldsymbol{x}\,d\boldsymbol{y}. \tag{30}$$

This equation provides a means to assess local dissipation at a specific point $\mathbf{x}$, by limiting the integral to $\mathbf{y}$ alone. It also empowers us to construct a purely dissipative model. This model is expressed as

$$\partial_j\tau_{ij}\simeq C_a^2\Delta^{-4}\int\left\{[\boldsymbol{u}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{y})]\cdot\nabla\zeta\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{\Delta}\right)\right\}_+[\boldsymbol{u}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{y})]\,d\boldsymbol{y}\,, \tag{31}$$

where $a_+ = max(0, a)$. Generally, ζ exhibits a reduction in value with increasing radius, and the turbulent viscosity tensor described by this model dissipates energy between locations compressed by the flow. In Cottet (1997), it is proved, showing correct asymptotic behavior close to the boundaries.

5. Conclusion

In summary, this study has undertaken a rigorous re-evaluation of the Smagorinsky model, providing valuable insights into the mathematical underpinnings of the subgrid-scale modeling within the context of asymptotic analysis applied to the LES model. The elucidation of this mathematical analysis not only stands as a foundational contribution but also opens avenues for an extensive exploration into the regularity of the Navier-Stokes Equations. We firmly believe that this investigation signifies a significant stride in advancing the Smagorinsky model. It culminates in the development of an anisotropic viscosity model for turbulent flows, rooted in the LES framework and featuring a robust mathematical formulation for anisotropic viscosity. This dedicated endeavor aims to deliver a comprehensive mathematical analysis, inspiring further inquiries and fostering a deeper comprehension of the intricacies surrounding the regularity of the Navier-Stokes equations.

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