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Mathematical analysis for simulation of incompressible fluid flow

Análise matemática para simulação de escoamento de fluidos incompressíveis

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Resumo
A omnipresença e a complexidade dos fenômenos físicos impulsionam a forte procura de ferramentas capazes de simulá-los, visto que, muitas aplicações requerem simulações computacionalmente viáveis, confiáveis e, preferencialmente, com baixo custo. Assim, a priori, este trabalho traz dois objetivos: i) compreender os fundamentos teóricos da dinâmica de fluidos com um importante método computacional destinado à simulação de escoamentos incompressíveis, denominado SPH (do inglês, Smoothed Particle Hydrodynamic), que a posteriori, será implementado; e ii) auxiliar na consolidação e aplicação de conceitos-chave em análise matemática para simulação computacional. Nossos esforços neste trabalho, fornecem fundamentos matemáticos, que por sua vez, descrevem a dinâmica do movimento fluidodinâmico.
Palavras-chave: Dinâmica dos Fluidos Computacional. Equações de Navier-Stokes. Simulação SPH.

Abstract
The omnipresence and complexity of physical phenomena drive the strong search for tools capable of simulating them, since many applications require computationally viable, reliable and, preferably, low-cost simulations. Thus, a priori, this work has two objectives: i) understand the theoretical foundations of fluid dynamics with an important computational method aimed at simulating incompressible flows, called SPH (Smoothed Particle Hydrodynamic), which will subsequently be implemented; and ii) assist in the consolidation and application of key concepts in mathematical analysis for computer simulation. Our efforts in this work provide mathematical foundations, which in turn, describe the dynamics of fluid dynamic motion.
Keywords: Computational Fluid Dynamics. Navier-Stokes Equations. SPH Simulation.

List of symbols and notations

In the vast realm of communication, symbols and notations serve as powerful tools, transcending linguistic barriers and conveying complex ideas with succinct precision. This compilation aims to elucidate a diverse range of notations and symbols, providing a key to unlocking embedded language. May this section act as a guiding compass for the reader, navigating them

through the symbolic landscapes of this work and enhancing their comprehension of the adopted formulations.

∇	Gradient operator
∂	Partial derivative
$\frac{Df}{Dt}$	Material derivative
div	Divergent
$\boldsymbol{u}$	Vector field
$\mathbf{T}$	Tensor field
λ, μ	Non-negative coefficients of viscosity

In each section of the text, various notations and their corresponding meanings are elucidated, providing a comprehensive understanding of the technical nuances within the respective domains.

1. Introduction

The prevalence and complexity of natural phenomena contribute to both the substantial demand for tools capable of simulating them and the challenges associated with designing such tools. In the realm of computer simulations, there is a pressing need to represent phenomena resulting from the interactions among natural elements, such as clouds, mountains, rivers, and trees. Frequently, these interactions either surpass the capabilities of real cameras, as seen in scenarios like space station explosions, or they prove prohibitively expensive and even perilous to film on location, exemplified by the flooding of a densely populated city or the rapid spread of a forest fire. A multitude of other applications also necessitates visually realistic computer simulations of natural phenomena, spanning domains like computer gaming and others.

The increase in computational power and the ready availability of ample memory have facilitated the simulation of a myriad of natural phenomena on standard hardware, with a particular emphasis on achieving visual realism. These advancements, coupled with the growing demand for simulations that closely mimic reality, have spurred the scientific community to focus on the development of methodologies and techniques for simulating physical phenomena. Consequently, there has been a substantial expansion in the body of literature dedicated to the simulation of natural elements over the past decade. It is imperative to acknowledge the necessity of crafting specialized algorithms and techniques for optimization in simulation, achieved through the adaptation of existing Computational Fluid Dynamics (CFD) methods and the creation of new techniques tailored to graphical requirements. This introductory work abstains from an exhaustive exploration of related issues, as these are effectively addressed through the application of the techniques adopted for simulating fluid motion across diverse applications.

2. Computer simulation

2.1 An overview of computer simulation

The fundamental objective of natural science is to provide the most precise description of reality possible, facilitating a comprehensive comprehension of natural phenomena and, in turn, obtaining a broader understanding of the behavior of objects under specific conditions. This pursuit encompasses the validation of a wide array of phenomena occurring on scales spanning orders of magnitude, ranging from the exploration of the fundamental nature of matter in quantum mechanics to the investigation of the origins of the universe. Historically, two methodological approaches have been employed to unveil the laws governing the natural world: the empirical and the theoretical. The empirical approach aims to uncover physical laws through observations supported by experiments and a variety of measuring devices and instruments. One of the prominent figures in

this regard was Galileo Galilei, who played a pivotal role in the scientific revolution. His groundbreaking experiments, such as those conducted from the Leaning Tower of Pisa, revealed that bodies of differing masses fall to the ground at the same rate, a foundational observation that significantly influenced the advancement of scientific understanding.

The theoretical approach, conversely, translates the laws of nature into mathematical expressions, utilizing the language of differential and integral calculus to elucidate how specific quantities evolve in relation to one another. A classic example of this theoretical insight is found in the well-known episode involving an apple's descent, which inspired Sir Isaac Newton to postulate that the same force of gravity governs the entire cosmos. This revelation led to the development of Newton's theory of gravitation, which also encompassed his three laws governing the motion of solid bodies. James Maxwell is credited with the formulation of the equations governing electromagnetic fields, and Albert Einstein is renowned for conceiving his groundbreaking theory of relativity, all through rigorous theoretical work. The mathematical foundation for describing fluid flow, encompassing variables such as velocity, pressure, and temperature in both spatial and temporal dimensions, relies on the Navier-Stokes and Euler equations. However, it is essential to acknowledge that both the theoretical and practical approaches have their limitations. In certain domains, conducting physical experiments, such as investigating the consequences of an oil spill or accidents in nuclear reactors, is inherently precluded due to safety concerns.

Experiments involving measurements frequently encounter limitations due to extended durations, which can be exceptionally protracted, or when the quantities under investigation exhibit extreme scales, ranging from the minute, such as electric currents in microprocessors, to the colossal, as observed in the birth of stars and galaxies. Moreover, a multitude of experiments entail diverse configurations and necessitate measurements at a limited number of discrete points. This is notably evident in fields like weather forecasting and the study of atmospheric phenomena, including tornadoes, hurricanes, and tsunamis. Conversely, the mathematical equations that accurately depict the physical world often become exceedingly intricate, rendering analytical solutions infeasible. In many instances, exact solutions can only be achieved for highly simplified models. In addition to the conventional practical and theoretical methodologies, simulations have emerged in recent years as a third approach that amalgamates the strengths of the two previously mentioned traditional methods.

At present, computational simulations find widespread utility across diverse scientific and industrial domains. Notably, in the field of mechanical engineering, these simulations play a pivotal role in the exploration of the properties of elastic solids. They are instrumental in vehicle design, structural stability analysis, and ongoing efforts to enhance structural integrity. In the realm of chemical applications, numerical simulations serve as a valuable tool for optimizing reactions involving various substances, as seen in processes like combustion. Additionally, their scope extends to the investigation of phenomena such as melting and coating processes, as well as the critical domain of weather forecasting. For a comprehensive overview of the numerous applications, one can refer to the work by Kaufmann & Smarr (1992). Those seeking a deeper understanding of the discretization of the differential equations involved can consult the research by Golub & Ortega (2014).

With its diverse array of applications, numerical simulation undeniably stands as a cutting-edge technological domain. In recent years, it has emerged as an invaluable tool for investigating environmental issues, among a myriad of others as previously mentioned. These are just a few tangible examples from a broader spectrum of knowledge that harnesses numerical simulation, prominently exemplified by Computational Fluid Dynamics (CFD). CFD, in particular, excels in providing highly accurate insights into the physical and physicochemical processes exhibited by fluid flow. This computational tool is underpinned by the Navier-Stokes equations, which, even in their two-dimensional formulation, incorporate non-linear terms, necessitating specific simplifications for problem resolution. In essence, these equations possess the capability to model a wide range of flow regimes, encompassing laminar, turbulent, compressible, and incompressible flows, both internally and externally. However, it is essential to recognize that recalibrating the flow

field around a body at each time step introduces an additional computational load, necessitating a stable and dependable numerical approximation for resolution. This exemplifies the critical relevance of this field of knowledge and its expanding horizons, underscoring its ever-increasing significance.

It's imperative to recognize that the current landscape offers a multitude of commercial software solutions designed to develop and implement algorithms, providing attractive and efficient alternatives for specific scenarios. Nevertheless, this landscape presents a dual dynamic, as some commercial software entails substantial costs and necessitates periodic license renewals. Additionally, many of these programs operate under the "closed-source" paradigm, restricting access to their internal programming structure, even when such modifications are warranted. This investigation does not aim to critique commercial software. On the contrary, these tools are robust and invaluable for evaluating diverse scenarios across a range of applications.

3. Equations of fluid dynamic motion

This section is dedicated to elucidating fundamental concepts and outcomes in mathematical fluid dynamics. The governing equations for fluid motion will be deduced based on fundamental conservation laws and the presumptions of continuum fluid mechanics. Our derivations will maintain generality until it becomes necessary to tailor our outcomes to the flows predominantly simulated in computer simulations, namely, incompressible inviscid flows and incompressible Newtonian flows characterized by uniform viscosity.

Through this theoretical exposition, we aim to acquire a robust comprehension of the underlying assumptions inherent in the equations of motion employed in physics-driven fluid simulation. We have selectively curated the prevailing concepts and findings from the existing literature, as well as, identified areas we perceive as ripe for further exploration in the advancement of simulations.

Our advancements stem from an in-depth exploration of select topics found in five seminal works Batchelor (1967), Chorin *et al*. (1990), and dos Santos (2007). For a more comprehensive understanding of mathematical fluid dynamics, we highly recommend interested readers to delve into these references for further elucidation.

Theorem 1 *(Reynolds' Transport Theorem).* Let $f: \overline{D} \times \mathrm{I} \longrightarrow \mathbb{R}$ as in the smoothness convection and $\Omega_t \coloneqq \varphi(\Omega_0, t) \subseteq D$, for each $t \in I$ and $\Omega_0 \subseteq D$ the (arbitrary) reference fluid region, then

$$\frac{d}{dt}\int\limits_{\Omega_t} f\, dV = \int\limits_{\Omega_t} \left(\frac{Df}{Dt} + f\, div\, \boldsymbol{u}\right) dV\,, \tag{1}$$

where dV denotes the volume element.

Prior to furnishing a proof for this outcome, we establish the subsequent lemma regarding the time derivative of $J(\boldsymbol{x}_0, t) \coloneqq \det\big(\nabla\varphi(\boldsymbol{x}_0, t)\big)$, where the Jacobian of φ is taken with respect to spatial coordinates.

Lemma 1. $\frac{\partial}{\partial t}J(\boldsymbol{x}_0, t) = J(\boldsymbol{x}_0, t)[div\, \boldsymbol{u}(\boldsymbol{x}_0, t), t]$

Proof of Lemma 1. Primarily, observe that

$$\frac{\partial}{\partial t}\nabla\varphi_i(\boldsymbol{x}_0,t) = \nabla\frac{\partial\varphi_i}{\partial t}(\boldsymbol{x}_0,t) = \nabla[\boldsymbol{u}_i(\varphi(\boldsymbol{x}_0,t),t)] = \nabla\boldsymbol{u}_i(\varphi(\boldsymbol{x}_0,t),t)\cdot\nabla\varphi(\boldsymbol{x}_0,t)$$
$$= \sum_{j=1}^{n}\frac{\partial\boldsymbol{u}_i}{\partial x_j}\,(\varphi(\boldsymbol{x}_0,t),t)\nabla\varphi_j(\boldsymbol{x}_0,t).$$

From this equality, the n-linearity and the antisymmetry of $det(\cdot)$, we have

$$\frac{\partial}{\partial t}\mathrm{J}(\boldsymbol{x}_0,t) = \frac{\partial}{\partial t}\det\big(\nabla\varphi\,(\boldsymbol{x}_0,t)\big) = \frac{\partial}{\partial t}\det\big(\nabla\varphi_1\,(\boldsymbol{x}_0,t),\dots,\nabla\varphi_n\,(\boldsymbol{x}_0,t)\big)$$
$$= \sum_{i=1}^{n} det\big(\nabla\varphi_1\,(\boldsymbol{x}_0,t),\dots,\nabla\varphi_i\,(\boldsymbol{x}_0,t),\dots,\nabla\varphi_n\,(\boldsymbol{x}_0,t)\big)$$
$$= \sum_{i=1}^{n} det\left(\dots,\sum_{i=1}^{n}\frac{\partial\boldsymbol{u}_i}{\partial x_j}\,(\varphi(\boldsymbol{x}_0,t),t)\,\nabla\varphi_j(\boldsymbol{x}_0,t),\dots\right) =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial\boldsymbol{u}_i}{\partial x_j}(\varphi(\boldsymbol{x}_0,t),t)\,det\big(\dots,\nabla\varphi_j\,(\boldsymbol{x}_0,t),\dots\big)$$
$$= \sum_{i=1}^{n}\frac{\partial\boldsymbol{u}_i}{\partial x_j}(\varphi(\boldsymbol{x}_0,t),t)\,det\big(\nabla\varphi_1\,(\boldsymbol{x}_0,t),\dots,\nabla\varphi_n\,(\boldsymbol{x}_0,t)\big)$$
$$= det\big(\nabla\varphi(\boldsymbol{x}_0,t)\big)\sum_{i=1}^{n}\frac{\partial\boldsymbol{u}_i}{\partial x_j}\,(\varphi(\boldsymbol{x}_0,t),t) = J(\boldsymbol{x}_0,t)[div\,\boldsymbol{u}\,\varphi(\boldsymbol{x}_0,t),t].\ \blacksquare$$

Proof of Reynolds' Transport Theorem. From the smoothness of $\varphi_{t_{|\Omega_0}}$ (and its inverse's), the continuity of $\varphi(\boldsymbol{x}_0,\ \cdot)$ and that $\varphi(\cdot,t_0)\equiv id(\cdot)$, we have the positivity of $J(\boldsymbol{x}_0,t)$ for each $x_0\in\Omega_0$ and each $t\in I$. From this, the preceding lemma and the change of variables theorem:

$$\frac{d}{dt}\int_{\Omega_t} f(\boldsymbol{x},t)\,dV = \frac{d}{dt}\int_{\varphi(\Omega_0,t)} f(\boldsymbol{x},t)\,dV$$

$$= \frac{d}{dt}\int_{\Omega_0} f(\varphi(\boldsymbol{x}_0,t),t)J(\boldsymbol{x}_0,t)dV$$

$$= \int_{\Omega_0}\left(\frac{\partial f}{\partial t} + \nabla f\cdot\frac{\partial \varphi}{\partial t}\right)(\varphi(\boldsymbol{x}_0,t),t)\,J(\boldsymbol{x}_0,t)\,dV$$

$$+ \int_{\Omega_0} f(\varphi(\boldsymbol{x}_0,t),t)\frac{\partial}{\partial t}J(\boldsymbol{x}_0,t)\,dV$$

$$= \int_{\Omega_0}\left(\frac{\partial f}{\partial t} + \nabla f\cdot\boldsymbol{u}\right)(\varphi(\boldsymbol{x}_0,t),t)\,J(\boldsymbol{x}_0,t)\,dV$$

$$+ \int_{\Omega_0} f(\varphi(\boldsymbol{x}_0,t),t)\,J(\boldsymbol{x}_0,t)\,[div\,\boldsymbol{u}\,(\varphi(\boldsymbol{x}_0,t),t)]\,dV$$

$$= \int_{\Omega_0}\frac{Df}{Dt}(\varphi(\boldsymbol{x}_0,t),t)\,J(\boldsymbol{x}_0,t)\,dV + \int_{\Omega_0}(f\,div\,\boldsymbol{u})(\varphi(\boldsymbol{x}_0,t),t)\,J(\boldsymbol{x}_0,t)\,dV$$

$$= \int_{\Omega_0}\left(\frac{Df}{Dt} + f div\,\boldsymbol{u}\right)(\varphi(\boldsymbol{x}_0,t),t)\,J(\boldsymbol{x}_0,t)\,dV$$

$$= \int_{\varphi(\Omega_0,t)}\left(\frac{Df}{Dt} + f div\,\boldsymbol{u}\right)(\boldsymbol{x},t)\,dV$$

$$= \int_{\Omega_t}\left(\frac{Df}{Dt} + f div\,\boldsymbol{u}\right)(\boldsymbol{x},t)\,dV\,. \quad \blacksquare$$

The ensuing corollary of the transport theorem establishes a link between the Lagrangian and Eulerian perspectives (in integral form).

Corollary 1. With Ω_t and f as in the transport theorem, let Ω_1 be the fixed set in $\mathbb{R}^n$ which coincides with Ω_t at $t = t_1$. Then at the (arbitrary) time t_1,

$$\frac{d}{dt}\int_{\Omega_t} f\,dV = \frac{\partial}{\partial t}\int_{\Omega_1} f\,dV + \int_{\partial\Omega_1} f\,\boldsymbol{u}\cdot\boldsymbol{n}\,dA\,,$$

where $\boldsymbol{n}$ is the unit outward normal, and dA the surface element, on $\partial\Omega_1$.

Proof. From the identity $div(f\boldsymbol{u}) = \nabla f\cdot\boldsymbol{u} + f\ div\ \boldsymbol{u}$, it suffices to note that $\frac{Df}{Dt} + f\ div\,\boldsymbol{u} = \frac{\partial f}{\partial t} + div\,(f\boldsymbol{u})$. From the transport and divergence theorems,

$$\frac{d}{dt}\int_{\Omega_t} f\,dV = \int_{\Omega_1}\left(\frac{Df}{Dt} + f\,div\,\boldsymbol{u}\right)dV = \int_{\Omega_1}\left(\frac{\partial f}{\partial t} + div(f\boldsymbol{u})\right)dV$$
$$= \int_{\Omega_1}\frac{\partial f}{\partial t}\,dV + \int_{\Omega_1} div(f\boldsymbol{u})\,dV = \frac{\partial}{\partial t}\int_{\Omega_1} f\,dV + \int_{\partial\Omega_1} f\,\boldsymbol{u}\cdot\boldsymbol{n}\,dA\,.\,\blacksquare \quad (2)$$

The left-hand side of this expression denotes the rate of change of the "f - content" of the fixed body of fluid occupying the region $\Omega_1 \subseteq D$ at time t_1. The first term on the right-hand side is the rate of change of the f - content of this fixed spatial domain. And the last term is the rate of outflow of f through the fixed boundary of Ω_1 (flux of f through $\partial\Omega_1$) - see more in Meyer (1982).

While real fluids undergo volume changes, a vast majority of fluid flows deemed 'significant' can be effectively modeled as incompressible flows with a high degree of precision. This implies that, for numerous practical applications, such as computation simulations of everyday fluids, we can confidently consider the incompressibility of the fluid flows in question.

Definition 1 *(Incompressible flow).* We say that φ denotes an incompressible flow *when*, for any fluid region $W \subseteq \Omega_0$ and every $t \in I$

$$volume\,(W) = volume\big(\varphi,(W,t)\big) \Leftrightarrow \int_{W} dV = \int_{\varphi(W,t)} dV \Leftrightarrow \frac{d}{dt}\int_{\varphi,(W,t)} dV = 0.$$

From this definition and the transport theorem, for incompressible flows,

$$0 = \frac{d}{dt}\int_{\varphi(W,t)} dV = \int_{\varphi(W,t)} div\,\boldsymbol{u}\,dv \Leftrightarrow div\,\boldsymbol{u} = 0,$$

where the right-hand side is often known as the incompressibility condition.

The incompressibility condition and *Lemma 1* also result that a flow is incompressible if and only if $J \equiv 1$, since $J(\cdot, t_0) = 1$. From the equation of continuity (the reader can check in Chorin *et al*. (1990)), and the fact that $\rho > 0$, a fluid is incompressible if and only if $\frac{D\rho}{Dt} = 0$, that is, the mass density is constant following the fluid. If the fluid is homogeneous (i.e., $\rho =$ constant in space), it also follows that the flow is incompressible if and only if ρ is constant in time. So, for a homogeneous incompressible fluid, $\rho(\boldsymbol{x}, t) = \rho_0 > 0$.

4. Smoothed Particle Hydrodynamics – SPH

This section is dedicated to providing an explanation of the Smoothed Particle Hydrodynamics (SPH) approach for simulating fluid flows. Our exposition is largely influenced by the contributions of Neto (2007), who expanded Smoothed Particle Hydrodynamics (SPH) in his PhD thesis, to model non-Newtonian viscoplastic and multiphase flows, particularly for applications in computer graphics.

We prioritize clarity and simplicity in our presentation. While our discussions are confined to a straightforward flow regime, we offer supplementary references for those interested in more comprehensive descriptions of the Smoothed Particle Hydrodynamics (SPH) structure and its computational implementation issues (refer to Gingold & Monaghan, 1977).

4.1 Flow regime and governing equations

Our explanation of the Smoothed Particle Hydrodynamics method for flow simulation is grounded in a scenario where the fluid exhibits uniform viscosity, and the pressure adheres to an equation of state of the following form $p(\boldsymbol{x},t) = f\big(\rho(\boldsymbol{x},t)\big)$. Given these assumptions, the governing equations for this fluid flow are:

$$\frac{D\rho}{Dt} = -\rho(\nabla \cdot \boldsymbol{u}) \tag{3}$$

$$\rho\frac{Du}{Dt} = -\nabla p + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) + \mu\Delta\boldsymbol{u} + \rho\boldsymbol{b} \tag{4}$$

known, respectively, as continuity and momentum. The Lagrangian formulation employed to articulate the governing equations stems from the discretization strategy embraced by Smoothed Particle Hydrodynamics (SPH), which also makes use of the identity $\boldsymbol{a} = \frac{D\boldsymbol{u}}{Dt}$ (where $\boldsymbol{a}$ stands for acceleration).

4.2 Representation of the field and discretization of the fluid

Instead of expressing field quantities through regularly sampled values and their differentials through difference equations (as done in stable fluids), SPH relies on scattered data approximation schemes and analytical differentiation of its approximations. This is achieved by leveraging the integral representation of a function $f\colon \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ as a convolution with Dirac's delta distribution δ and that this distribution can be characterized as a generalized limit of specific smooth functions W_h, i.e., $h \longrightarrow 0 \Longrightarrow W_h \longrightarrow \delta$:

$$f(\boldsymbol{x}) = \int_{\Omega} f(\boldsymbol{x}')\,\delta\,(\boldsymbol{x}-\boldsymbol{x}')\,d\boldsymbol{x}' = \lim_{h\longrightarrow 0}\left\{\int_{\Omega} f(\boldsymbol{x}')\,W_h\,(\boldsymbol{x}-\boldsymbol{x}')\,d\boldsymbol{x}'\right\}. \tag{5}$$

Motivated by Eq. (4), the SPH approximation $\langle f\rangle$ to the field f is defined by a given family W_h of smooth kernel functions and a fixed $h > 0$ as

$$\langle f(x)\rangle := \int_{\Omega} f(\boldsymbol{x}')\,W_h\,(\boldsymbol{x}-\boldsymbol{x}')\,d\boldsymbol{x}'\,, \tag{6}$$

which is numerically discretized by the quadrature formula

$$\langle f(x)\rangle \approx \sum_j f\big(\boldsymbol{x}_j\big)\,W_h\,\big(x-x_j\big)\,\Delta V_j = \sum_j f_j\,\frac{m_j}{\rho_j}\,W_h\big(x-x_j\big)\,, \tag{7}$$

where the weight $\Delta V_j = \frac{m_j}{\rho_j}$ corresponds to the volume associated to the $j-th$ fluid particle (located at the quadrature point $\boldsymbol{x}_j \in \Omega$), m_j to its mass and ρ_j to its associated specific mass. This quadrature formulation for the integral SPH approximation is construed as discretizing the fluid mass into a finite number of particles that evolve based on the governing equations. These equations describe the system dynamics through a coupled set of nonlinear ordinary differential equations (ODEs), a topic addressed in the subsequent subsection.

Inspecting Eq. (6), we notice that whenever the family W_h is composed by compactly supported kernels, with influence radius say κh, i.e., $\exists \kappa > 0$ such that $\|\boldsymbol{x} - \boldsymbol{x}_j\| \geq \kappa h \Longrightarrow W_h(\boldsymbol{x} - \boldsymbol{x}_j) = 0$, the sum in Eq. (6) effectively only takes place for those particles which are less than κh away from $\boldsymbol{x}$. Defining the neighboring particles of a point $\boldsymbol{x} \in W$ as $N(\boldsymbol{x}) := \{j \in \mathbb{N} \mid \|\boldsymbol{x} - \boldsymbol{x}_j\| < \kappa h\}$, we can rewrite the discrete SPH approximation Eq. (6) as

$$\langle f(x) \rangle \approx \sum_j f_j \frac{m_j}{\rho_j} W_h(x - x_j) = \sum_{j \in N(\boldsymbol{x})} f_j \frac{m_j}{\rho_j} W_h(x - x_j) . \tag{8}$$

4.3 The discretized equations that govern the system

The method utilized by SPH to discretize the governing equations relies on the Lagrangian formulation of the fluid flow's governing equations. While Müller *et al*. (2003) deviates from the mentioned equations, we embrace the differential expression of momentum along with the tensor field, namely:

$$\rho \frac{D\boldsymbol{u}}{Dt} = div\, \boldsymbol{T} + \rho \boldsymbol{b}, \tag{9}$$

where $\boldsymbol{T} = -pI + \boldsymbol{\sigma}$, and $\boldsymbol{\sigma} = \lambda\, tr(\boldsymbol{D})\boldsymbol{I} + 2\mu \boldsymbol{D}$; yet $D = {}^1/_2\,(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\boldsymbol{T}})$, we have the following set of equations:

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho}\, div\, [\lambda\, tr(D) I + 2\mu D] + \boldsymbol{b} . \tag{10}$$

As we have discretized the fluid mass into a finite number of particles, the behavior of this particle system is determined by the temporal changes in the position, velocity, and density of each particle. Hence, the behaviour of our system is described by how $\frac{d\boldsymbol{x}}{dt}$, $\frac{D\rho}{Dt}$ and $\frac{D\boldsymbol{u}}{Dt}$ are determined for each particle along this particle's trajectory $\boldsymbol{x}(t)$. Thus, for particle i, we have:

$$\frac{d\boldsymbol{x_i}}{dt} = \boldsymbol{u}_i , \tag{11}$$

$$\frac{d\rho_i}{dt} = -\rho_i \langle \nabla \cdot \boldsymbol{u}_i \rangle , \tag{12}$$

$$\boldsymbol{D}_i = \frac{1}{2}(\langle \nabla \boldsymbol{u}_i \rangle + \langle \nabla \boldsymbol{u}_i \rangle^{\boldsymbol{T}}) , \tag{13}$$

$$p_i = f(\rho_i) , \tag{14}$$

$$\boldsymbol{\sigma}_i = \lambda\, tr(\boldsymbol{D}_i)\boldsymbol{I} + 2\mu \boldsymbol{D}_i , \tag{15}$$

$$\frac{d\boldsymbol{u}_i}{dt} = -\langle\frac{1}{\rho_i}\nabla p_i\rangle + \langle\frac{1}{\rho_i} div\ \boldsymbol{\sigma}_i\rangle + \langle\boldsymbol{b}_i\rangle\,. \tag{16}$$

Observe the transition in notation from $\frac{D}{Dt}$ to $\frac{d}{dt}$, as we are addressing the variations in a particle's attributes rather than derivatives of field quantities along trajectories in this context. Where each spatial derivative is approximated by the symmetric rules we derived before (to ensure conformance with Newton's third law).

5. Conclusion

Thinking in the computational context, challenges often arise when dealing with boundaries. Although in principle no specific computational implementation was carried out in this study, it is recognized that extremely small-time steps are imperative for numerical simulations in order to avoid what are commonly known as 'numerical explosions', or, in other words, numerical inconsistencies. Consequently, a subsequent study will be presented, based on the works of dos Santos & de Oliveira Sales (2023), which will present a new numerical approach that aims to treat these numerical inconsistencies, and how they can be overcome with the objective of optimizing the calculation process, as well as, its computational cost. In principle, this work employs a numerical scheme for responding to readings and collisions between fluid dynamic particles, which are similar to what occurs in rigid body simulations. The initial exploration facilitated the adoption of mathematical procedures that prove to be effective in improving the stability of particles in the flow acceleration field during their evolution. Each particle operates independently, using interpolated accelerations. This mathematical approach demonstrates adaptability in both temporal and spatial domains. Initial studies, exemplified by Hernquist & Katz (1989), report the first advances in relation to the computational process (simulation) and the complexity in advancing simulation time.

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