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A mathematical analysis to the approximate weak solution of the Smagorinsky Model for different flow regimes

Uma análise matemática para a solução fraca aproximada do Modelo Smagorinsky para diferentes regimes de fluxo

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Resumo

Este estudo investiga a aproximação numérica de equações não estacionárias de Navier-Stokes em regimes turbulentos, empregando o Modelo Smagorinsky (SM). Ao tratar o modelo como inerentemente discreto, implementamos uma discretização temporal semi-implícita usando o método de Euler. Esta abordagem inclui análises abrangentes de estabilidade, aplicáveis a um espectro de regimes de fluxo, e uma exploração da dinâmica assintótica do balanço de energia durante movimentos de fluidos. A principal contribuição deste estudo encontra-se na sua abordagem metódica à aproximação numérica de equações não estacionárias de Navier-Stokes dentro de regimes turbulentos utilizando o Modelo Smagorinsky (SM). A adoção de uma discretização temporal semi-implícita com o método de Euler, aliada a uma análise meticulosa do balanço de energia, estabelece uma base robusta e adaptável a diversas condições de fluxo.

Palavras-chave: Modelo de Smagorinsky. Solução Fraca. Equações de Navier–Stokes. Balanço Assintótico.

Abstract

This study delves into the numerical approximation of non-stationary Navier-Stokes equations within turbulent regimes, employing the Smagorinsky Model (SM). By treating the model as inherently discrete, we implement a semi-implicit time discretization using the Euler method. This approach includes comprehensive stability analyses, applicable to a spectrum of flow regimes, and an exploration of the asymptotic energy balance dynamics during fluid movements. The primary contribution of this study is found in its methodical approach to the numerical approximation of non-stationary Navier-Stokes equations within turbulent regimes using the Smagorinsky Model (SM). The adoption of a semi-implicit time discretization with the Euler method, coupled with a

meticulous analysis of energy balance, establishes a robust foundation adaptable to diverse flow conditions.

Keywords: Smagorinsky model. Weak Solution. Navier–Stokes equations. Asymptotic Balance.

1. Introduction

The numerical approximation of the unstable Navier-Stokes equations poses various technical challenges. Addressing the incompressibility constraint during time discretization necessitates specific techniques to ensure the stability of pressure discretization. Furthermore, achieving stability in velocity discretization mandates the use of implicit or semi-implicit discretization methods for the nonlinear convection term, eliminating the need for excessively small-time steps. To attain high accuracy, it is crucial to employ high-order solvers designed to meet the aforementioned stability requirements while maintaining relatively low computational complexity.

Direct approaches involve integrating time discretizations extrapolated from standard methods for solving ordinary differential equations into the Navier-Stokes equations. This includes employing Euler's implicit and explicit temporal discretizations, as well as Crank-Nicolson, harmonized with mixed spatial discretizations. Further refinement includes extrapolating the construction structure of Runge-Kutta methods, leading to fractional step methods, as investigated by scholars such as Heywood & Rannacher (1990) and Temam (1977). A comprehensive examination of fractional step methods is available in Gresho and Sani (2000), and the use of Crank-Nicolson and fractional step scheme discretizations for solving incompressible Navier-Stokes equations is detailed in Turek's book (1999). It is important to note that these techniques are primarily designed for flows where diffusion plays a dominant role. As the Reynolds number increases, the convection term gains dominance, leading to instabilities in the discrete equations, particularly with higher Reynolds numbers. The Characteristic Method, a solution to this problem, is not discussed here and relies on a temporal discretization that transforms the material derivative into a temporal derivative along flow lines.

All referenced studies pertain to the discretization of the Smagorinsky Model. However, challenges arise from nonlinearities and the representation of submesh effects. Employing high-precision methods becomes imperative to mitigate errors from numerical discretization. In some experiments, submesh model impacts are obscured by numerical errors when using accurate second-order methods. Analyses based on the similarity hypothesis suggest that achieving negligible numerical diffusion relative to turbulent diffusion requires an eighth order of precision (refer to Sagaut (2002), Chapter 8). In practical terms, it is noteworthy that second-order methods are influenced by the choice of the submesh model, as illustrated in benchmark tests for laminar flow problems in Turek (1999) and numerical solutions of Large Eddy Simulation (LES) models by John (2006). Additionally, meticulous execution of the temporal discretization of the "laws of the wall" is necessary to maintain dissipative characteristics and stability.

Both criteria are satisfied by the implicit method, as discussed here; however, implicit discretizations result in algebraic systems of nonlinear equations requiring specialized solvers. The conventional numerical analysis of these discretization methods for the unsteady Navier–Stokes equations demonstrates their stability under "natural" $L^2\big((0,T),\boldsymbol{H}^1(\Omega)\big)$ and $L^\infty\big((0,T),\boldsymbol{L}^2(\Omega)\big)$ norms. This guarantees a weak convergence of numerical approximations for a weak solution of the Navier-Stokes equations. However, the limited regularity of the weak solution makes it difficult to demonstrate strong convergence in numerical approximations and restricts the energy balance to an inequality. For the Smagorinsky Model (SM), these challenges are aggravated by the existence of nonlinearities arising from the eddy diffusion term in the SM and the wall-law term. In this context, our approach involves presenting a weak formulation for SM. We propose a scheme based on the semi-implicit Euler method for temporal advancement. In this study, we refrain from incorporating any velocity-pressure decoupling strategy to avoid unnecessary complexities. The mathematical analysis employed, with due diligence, can be extended to more general discretizations, such as the Crank-Nicolson scheme or Fractional Step methods. It is worth mentioning that the application of alternative discretization methods, at an opportune moment, requires the integration of techniques adapted to the terms of eddy diffusion and wall law, this can be verified in the work of Santos *et al.* (2018).

This work is organized as follows: In Section 2, we present the weak formulation. We establish that the smooth solutions derived from this weak formulation. Section 3 is dedicated to outlining the asymptotic energy, we present a proof that presents a unique solution that satisfies the asymptotic escapes. In Section 4, an analysis is presented that can be extended to the Euler method approximation of the LES-Smagorinsky model. Finally, in the Section 5, the completion of this work.

2. Formulation of the Smagorinsky Model in Weak Form

In our analysis we prove the stability of the implicit Euler discretization of SM in $L^2\big((0,T),\boldsymbol{H}^1(\Omega)\big)$, and $L^\infty\big((0,T),\boldsymbol{L}^2(\Omega)\big)$ norms and the weak convergence in these spaces to a weak solution of the Navier–Stokes equations including wall laws:

$$\begin{cases} \partial_t \boldsymbol{v} + \nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{v}) - \nabla\cdot(\nu\, D\boldsymbol{v}) + \nabla \mathrm{p} = \mathbf{f} & in \quad \Omega\times(0,T); \\ \nabla\cdot\boldsymbol{v} = 0 & in \quad \Omega\times(0,T); \\ [\nu\, D\boldsymbol{v}\cdot\boldsymbol{n}]_\tau = g(\boldsymbol{v})_\tau & on \quad \Gamma_n\times(0,T); \\ \boldsymbol{v}\cdot\boldsymbol{n} & on \quad \Gamma_n\times(0,T); \\ \boldsymbol{v} = 0 & on \quad \Gamma_D\times(0,T); \\ \boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{v}_0(\boldsymbol{x}) & in \quad \Omega. \end{cases} \tag{2.1}$$

The influence of eddy diffusion on the large-scale flow diminishes weakly when all scales are resolved. Our analysis relies on the compactness method. We derive estimates for the time and space derivatives of velocity to converge to the limit in the nonlinear, viscosity (laminar and eddy), and wall-law terms. We subsequently derive estimates for the primitive in time of the pressure in $L^\infty\big((0,T),\boldsymbol{L}^2(\Omega)\big)$. The examination of enhanced regularity in weak solutions of Navier–Stokes equations is grounded in the utilization of specific test functions that nonlinearly depend on velocity. As of the authors' knowledge, the extension of this analysis to numerical discretizations has not been undertaken.

We will exclusively establish weak convergence for a solution of the Navier-Stokes equations. The estimates we derive, akin to the conventional analysis of the Navier-Stokes equations, fall short of providing a solution smooth enough to serve as a test function in the variational formulation. Therefore, in principle, demonstrating strong convergence is unattainable. This limitation is a prevailing drawback in the current state of research, significantly impacting the analysis of the Navier-Stokes equations. In our analysis, we conduct an error estimation applicable to general flow regimes, not restricted solely to convection-dominated flows, as observed in the stationary case. In this study, similar to the steady state, we will demonstrate that the convergence order is suboptimal concerning the precision of the finite element discretization, attributed to the presence of the eddy viscosity term. The appropriate positioning of discrete problems was also investigated. We establish that each genuine discrete problem is well-posed. However, uniform continuity concerning the discretization parameters would only be valid if the discrete solutions were confined within $L^2\big((0,T),\boldsymbol{W}^3(\Omega)\big)$.

In this section, we introduce a variational formulation for the mixed boundary value problem (2.1) associated with the Navier-Stokes equations. In Section 3, we will proceed to approximate this formulation using the Smagorinsky Method (SM) with finite elements. An existing technical difficulty in analyzing the Navier-Stokes instability and related equations is obtaining estimates of the pressure in $L^p(Q)$ norms, where we recall that $Q = \Omega \times (0,T)$. Typically, in the context of the continuous problem, this is achieved through the utilization of test functions characterized by a nonlinear dependence on the pressure. Nonetheless, our focus here revolves around the approximation of the pressure, given its significant physical influence in numerous practical applications. To address these challenges, we will surmount them by substituting the pressure with its temporal primitive as an unknown variable. We will demonstrate in a fairly intuitive manner that this temporal primitive of the pressure possesses $L^\infty\big((0,T),\ \boldsymbol{L}^2(\Omega)\big)$ regularity.

To express the weak formulation of the Eq. (2.1), let's introduce the space of divergence-free functions:

$$\boldsymbol{W}_{Div}(\Omega) = \{\boldsymbol{w} \in \boldsymbol{W}_D(\Omega)\ s.t.\ \nabla\cdot\boldsymbol{w} = 0\ \ a.e.in\ Q\}.$$

The space $\boldsymbol{W}_{Div}(\Omega)$ is a closed subspace of $\boldsymbol{W}_D(\Omega)$, and then it is a Hilbert space endowed with the $\boldsymbol{H}^1(\Omega)$ norm.

Definition 2.1. Let $\boldsymbol{f} \in L^2(\boldsymbol{W}_D(\Omega)'), \boldsymbol{v}_0 \in W_D(\Omega)'$. A pair $(\boldsymbol{v},p) \in \mathcal{D}'(Q)^d \times \mathcal{D}'(Q)$ is a weak solution of the Navier–Stokes problem (2.1) if for all $\boldsymbol{v} \in L^2\big(\boldsymbol{W}_{Div}(\Omega)\big) \cap L^\infty(\boldsymbol{L}^2)$, there exists $P \in L^2(L^2)$ such that $p = \partial_t P$, and for all $w \in W_D(\Omega), \varphi \in \mathcal{D}([0,T])$ such that $\varphi(T) = 0$:

$$\begin{cases} -\displaystyle\int_0^T (\boldsymbol{v}(t),\boldsymbol{w})_\Omega \varphi'(t)dt - \langle \boldsymbol{v}_0,\boldsymbol{w}\rangle\varphi(0) \\ +\displaystyle\int_0^T \big[b(\boldsymbol{v}(t);\boldsymbol{v}(t),\boldsymbol{w})dt + a(\boldsymbol{v}(t),\boldsymbol{w}) + \langle G\big(\boldsymbol{v}(t)\big),\boldsymbol{w}\rangle\big]\varphi(t)dt \\ +\displaystyle\int_0^T (P(t),\nabla\cdot\boldsymbol{w})_\Omega \varphi'(t)dt = \int_0^T \langle \boldsymbol{f}(t),\boldsymbol{w}\rangle\, \varphi(t)dt. \end{cases} \tag{2.2}$$

This definition is meaningful because, given the required regularity for $\boldsymbol{v}$ and P, all terms in (2.2) are integrable over the interval $(0;T)$. The solutions satisfying this definition weakly fulfill the Navier–Stokes equations in the subsequent manner.

Lemma 2.1. Let $(\boldsymbol{v},p) \in \mathcal{D}'(Q)^d \times \mathcal{D}'(Q)$ be a weak solution of the Navier-Stokes Eq. (2.1). Then

(i) *The equations*

$$\begin{aligned} &\partial_t\boldsymbol{v} + \nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{v}) - \nabla\cdot(\nu\, D\boldsymbol{v}) + \nabla p = \boldsymbol{f} \\ &\nabla\cdot\boldsymbol{v} = 0 \end{aligned} \tag{2.3}$$

respectively hold in $\mathcal{D}'(Q)$ and in $\boldsymbol{L}^2(Q)$.

(ii)

$$\boldsymbol{v} \in \boldsymbol{C}^0([0,T], \boldsymbol{W}_D(\Omega)') \text{ and } \boldsymbol{v}(0) = \boldsymbol{v}_0 \text{ in } \boldsymbol{W}_D(\Omega)'.$$

(iii)

$$\gamma_0 \boldsymbol{v} = 0 \; in \; L^2\left(\boldsymbol{H}^{\frac{1}{2}}(\Gamma_D)\right), \;\; \gamma_n \boldsymbol{v} = 0 \; in \; L^2\big(\boldsymbol{L}^4(\Gamma)\big).$$

(iv) *If* $\boldsymbol{v} \in L^2(\boldsymbol{H}^2)$, $\partial_t \boldsymbol{v} \in L^2(\boldsymbol{L}^2)$, *and* $p \in L^2(\boldsymbol{H}^1)$, *then*

$$-[\nu \cdot D\boldsymbol{v} \cdot \boldsymbol{n}]_\tau = g(\boldsymbol{v})_\tau \;\; in \; L^1\left(L^{3/2}(\Gamma_n)\right)^{d-1}.$$

Proof.

(i) As $\boldsymbol{v} \in L^1(Q)$, then $\boldsymbol{v}$ generates a distribution, and

$$\langle \partial_t \boldsymbol{v}, \boldsymbol{w} \otimes \varphi \rangle_{\mathcal{D}(Q)} = -\int_Q \boldsymbol{v}(\boldsymbol{x},t)\partial_t\big(\boldsymbol{w}(\boldsymbol{x})\varphi(t)\big)d\boldsymbol{x}dt = -\int_0^T (\boldsymbol{v}(t), \boldsymbol{w})_\Omega \, \varphi'(t)dt\,,$$

for all $\boldsymbol{w} \in \mathcal{D}(\Omega)^d$, $\varphi \in \mathcal{D}(0,T)$. Similarly, as $P \in L^1(Q)$,

$$\langle \nabla(\partial_t P), \boldsymbol{w} \otimes \varphi \rangle_{\mathcal{D}(Q)} = \int_0^T (P(t), \nabla \cdot \boldsymbol{w})_\Omega \, \varphi'(t)dt\,,$$

Then, integrating by parts and using $\langle G\big(\boldsymbol{v}(t)\big), \boldsymbol{w}\rangle = 0$ and $\nabla \cdot \boldsymbol{v} = 0$ *a.e.* in Q, (2.2) implies

$$\langle \partial_t \boldsymbol{v} + \nabla \cdot (\boldsymbol{v} \otimes \boldsymbol{v}) - \nabla \cdot (\nu \, D\boldsymbol{v}) + \nabla p - \boldsymbol{f}, \boldsymbol{w} \otimes \varphi \rangle_{\mathcal{D}(Q)} = 0,$$

for all $\boldsymbol{w} \in \mathcal{D}(\Omega)^d$, $\varphi \in \mathcal{D}(0,T)$. Therefore, we deduce (2.3). Also, as $\boldsymbol{v} \in L^2\big(\boldsymbol{W}_{Div}(\Omega)\big)$, then $\nabla \cdot \boldsymbol{v} = 0$ in $L^2(Q)$.

(ii) Let $\Phi(t) \in \boldsymbol{W}_D(\Omega)'$ defined a.e. in $(0,T)$ by

$$\langle \Phi(t), \boldsymbol{z} \rangle = b(\boldsymbol{v}(t); \boldsymbol{v}(t), \boldsymbol{z}) + a(\boldsymbol{v}(t), \boldsymbol{z}) + \langle G\big(\boldsymbol{v}(t)\big), \boldsymbol{z}\rangle - \langle \boldsymbol{f}(t), \boldsymbol{z}\rangle\,.$$

Exists a constant $\boldsymbol{C} > 0$ such that,

$$\|\Phi(t)\|_{\boldsymbol{W}_D(\Omega)'} \leq \boldsymbol{C}\left(\big\|D\big(\boldsymbol{v}(t)\big)\big\|_{0,2,\Omega}^2 + \big\|D\big(\boldsymbol{v}(t)\big)\big\|_{0,2,\Omega} + \|\boldsymbol{f}(t)\|_{\boldsymbol{W}_D(\Omega)}\right).$$

Then, $\Phi \in L^1(\boldsymbol{W}_D(\Omega)')$. From (2.2), we deduce that for all $\boldsymbol{W}_{Div}(\Omega)$, $\varphi \in \mathcal{D}(0,T)$,

$$\int_0^T \langle \boldsymbol{v}(t), \boldsymbol{w}\rangle_{\boldsymbol{W}_{Div}(\Omega)} \, \varphi'(t)dt = \int_0^T \langle \Phi(t), \boldsymbol{w}\rangle_{\boldsymbol{W}_{Div}(\Omega)} \varphi(t)dt\,.$$

According to the works of Santos & Sales (2023) and Santos & Silva (2023), with regards to the functional Banach space's, the following lemma is valid:

Lemma 2.2. Let E be a Banach space. Let $v, g \in L^1(0,T;E)$. Then the following three conditions are equivalent:

(i) *f is a.e. in $(0,T)$ equal to a primitive of g, i.e.,*

$$f(t) = \xi + \int_0^T g(s)\,ds, \;\; a.e.\ in\ (0,T),\;\; for\ some\ \xi \in E.$$

(ii) *For each $\varphi \in \mathcal{D}(0,T)$,*

$$\int_0^T f(t)\,\varphi'(t)\,dt = -\int_0^T g(t)\,\varphi(t)\,dt\,.$$

(iii) *For each $\eta \in E'$,*

$$\frac{d}{dt}\langle f,\eta\rangle_{E'} = \langle g,\eta\rangle_{E'} \;\; in\ \mathcal{D}'(0,T).$$

If these conditions are satisfied, then f is a.e. in $(0,\mathrm{T})$ equal to a function of $\boldsymbol{C}^0([0,T],E)$, and we set $g = \partial_t f$.

So, by *Lemma 2.2,* *(ii)* this implies $\partial_t \boldsymbol{v} = -\Phi \in L^1(\boldsymbol{W}_{Div}(\Omega)')$, and $\boldsymbol{v} \in \boldsymbol{C}^0([0,T],\boldsymbol{W}_{Div}(\Omega)')$. Also, by *Lemma 2.2, (iii), if* $\varphi \in \mathcal{D}([0,T])$ is such that $\varphi(T) = 0$, then

$$\int_0^T \langle \partial_t \boldsymbol{v}(t), \boldsymbol{w}\rangle_{\boldsymbol{W}_{Div}(\Omega)}\varphi(t)\,dt = -\langle \boldsymbol{v}(0),\boldsymbol{w}\rangle_{\boldsymbol{W}_{Div}(\Omega)}\,\varphi(0) - \int_0^T (\boldsymbol{v}(t),\boldsymbol{w})_\Omega\,\varphi'(t)dt\,.$$

As $\boldsymbol{v}_0 \in \boldsymbol{W}_D(\Omega)' \hookrightarrow \boldsymbol{W}_{Div}(\Omega)'$, by (2.2), it follows:

$$\int_0^T \langle \partial_t \boldsymbol{v}(t) + \Phi(t), \boldsymbol{w}\rangle_{\boldsymbol{W}_{Div}(\Omega)}\,\varphi(t)\,dt + \langle \boldsymbol{v}_0 - \boldsymbol{v}(0), \boldsymbol{w}\rangle_{\boldsymbol{W}_{Div}(\Omega)}\,\varphi(0) = 0\,,$$

and so $\langle \boldsymbol{v}_0 - \boldsymbol{v}(0), \boldsymbol{w}\rangle_{\boldsymbol{W}_{Div}(\Omega)} = 0$ for all $\boldsymbol{w} \in \boldsymbol{W}_{Div}(\Omega)$. We conclude that $\boldsymbol{v}(0) = \boldsymbol{v}_0$ in $\boldsymbol{W}_{Div}(\Omega)'$.

(iii) As γ_0 is a linear mapping from $\boldsymbol{H}^1(\Omega)$ to $\boldsymbol{H}^{1/2}(\Gamma_D)$, then $\gamma_0\boldsymbol{v} \in L^2\left(H^{1/2}(\Gamma_D)\right)$. As $\boldsymbol{v} \in L^2\big(\boldsymbol{W}_{Div}(\Omega)\big)$, then $\gamma_0\boldsymbol{v} = 0$ in $L^2\left(\boldsymbol{H}^{1/2}(\Gamma_D)\right)$. Similarly, $\gamma_n\boldsymbol{v} = 0$ in $L^2\big(\boldsymbol{L}^4(\Gamma_n)\big)$.

(iv) Assume $\boldsymbol{v} \in L^2(\boldsymbol{H}^2)$, $\partial_t\boldsymbol{v} \in L^2(\boldsymbol{L}^2)$, $p \in L^2(\boldsymbol{H}^1)$. Applying Green's formula in item (iv) of *Lemma 2.2*, we obtain:

$$\int_0^T \int_{\Gamma_n} [\nu\, D\, \boldsymbol{v}(\boldsymbol{x},t)\cdot \boldsymbol{n}(\boldsymbol{x}) - g(\boldsymbol{v}(\boldsymbol{x},t))]_\tau \cdot w_\tau(\boldsymbol{x})\varphi(t)\, d\Gamma_n(\boldsymbol{x})\, dt = 0\,, \tag{2.4}$$

for all $\boldsymbol{w} \in \boldsymbol{W}_D(\Omega), \varphi \in \mathcal{D}(0,T)$. As $g(v) \in L^1\left(\boldsymbol{L}^{3/2}(\Gamma_n)\right)$. As $\gamma_0(D\boldsymbol{v}) \in L^2\left(\boldsymbol{H}^{1/2}(\Gamma)\right)$, then $[\nu\, D\boldsymbol{v}\cdot\boldsymbol{n} - g(\boldsymbol{v})] \in L^1\left(\boldsymbol{L}^{3/2}(\Gamma_n)\right)$. Consequently, by *Lemma 2.1* implies $[\nu\, D\boldsymbol{v}\cdot\boldsymbol{n} - g(\boldsymbol{v})]_\tau = 0$ in $L^1\left(L^{3/2}(\Gamma_n)^{d-1}\right)$.

With some more technical work it is possible to prove that $\boldsymbol{v}$ is weakly continuous from $[0,T]$ into $\boldsymbol{L}^2(\Omega)$, i.e., the functions $t \in [0,T] \longmapsto (\boldsymbol{v}(t),\boldsymbol{w})_\Omega$ are continuous, for any $\boldsymbol{w} \in \boldsymbol{L}^2(\Omega)$. Then, the initial condition $\boldsymbol{v}(0) = \boldsymbol{v}_0$ holds in $\boldsymbol{L}^2(\Omega)$.

3. Asymptotic Energy

In the context of evolution, there is a lack of evidence supporting the assertion that weak solutions of the Smagorinsky model satisfy an asymptotic energy identity, akin to what is observed in the case of the steady Smagorinsky model, as far as the authors are aware up to the present time. This absence of proof stems from the limited regularity of the weak solution, leading to significant ramifications. Specifically, the dissipated energy caused by eddy diffusion cannot be demonstrated to approach zero in the limit. Furthermore, transitioning to the limit in the term corresponding to the energy dissipated at the wall is not feasible. Additionally, the weak solution cannot serve as a test function in the weak formulation (2.2). Consequently, even without turbulence modeling, establishing strong convergence remains unattainable. However, an alternative can be pursued: the proof of an energy inequality, linked to the dissipative nature, for certain simplified wall laws. To illustrate, let us consider the Glaucker–Manning law as the prescribed wall law, (see more in Gauckler (1867) and Manning (1891)):

$$g(\boldsymbol{v}) = c_f\, |\boldsymbol{v}|\, \boldsymbol{v},$$

where $c_f > 0$ is a friction coefficient. Then the following holds:

Lemma 3.1. Let $\boldsymbol{v} \in L^2(\boldsymbol{W}_{Div}(\Omega) \cap L^\infty(\boldsymbol{L}^2))$ a weak solution (together with some pressure $p \in \mathcal{D}'(Q)$) of problem (2.2) that is obtained as a weak limit of some sequence $(\boldsymbol{v}_h)_{h>0}$. So, we have:

$$\begin{aligned}\frac{1}{2}\|\boldsymbol{v}(t)\|_{0,2,\Omega}^2 + \nu \int_0^t \|D(\boldsymbol{v}(s))\|_{0,2,\Omega}^2\, ds \;+ \int_0^t \int_{\Gamma_n} \langle G(\boldsymbol{v}(s)), \boldsymbol{v}(s)\rangle\, ds \\ \le \frac{1}{2}\|\boldsymbol{v}(t)\|_{0,2,\Omega}^2 + \int_0^t \langle \boldsymbol{f}(s), \boldsymbol{v}(s)\rangle\, ds\,,\end{aligned} \tag{3.1}$$

for almost every $[0,T]$.

Proof. We start from estimate (3.2). Using that

$$\langle \boldsymbol{f}^{n+1}, \boldsymbol{v}_h^{n+1}\rangle = \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \langle \boldsymbol{f}(t), \widetilde{\boldsymbol{v}}_h(t)\rangle\, dt\,,$$

we deduce,

$$\frac{1}{2}\|\widetilde{\boldsymbol{v}}_h(t)\|_{0,2,\Omega}^2 + \nu\int_0^t \left\|D\big(\widetilde{\boldsymbol{v}}_h(s)\big)\right\|_{0,2,\Omega}^2 ds \quad + \int_{t_n}^t \langle G\big(\widetilde{\boldsymbol{v}}_h(s)\big), \widetilde{\boldsymbol{v}}_h(s)\rangle\, ds$$
$$\leq \frac{1}{2}\|\boldsymbol{v}_h^{n+1}\|_{0,2,\Omega}^2 + \nu\,\Delta t\,\|D(\boldsymbol{v}_h^{n+1})\|_{0,2,\Omega}^2 + \Delta t\,\langle G(\boldsymbol{v}_h^{n+1}), \boldsymbol{v}_h^{n+1}\rangle \tag{3.2}$$
$$\leq \frac{1}{2}\|\boldsymbol{v}_h^{n}\|_{0,2,\Omega}^2 + \int_{t_n}^{t_{n+1}} \langle \boldsymbol{f}(s), \tilde{v}_h(s)\rangle\; ds\,,$$

for all $t \in (t_n, t_{n+1})$.

Then, summing up in n from $n = 0$ to $n = k - 1$, for $k = 2, \ldots, N$, and using the extraction of convergent subsequences (cf. Brézis (1983)), i.e., $\|\boldsymbol{v}_{0h}\|_{\boldsymbol{L}^2(\Omega)} \leq \|\boldsymbol{v}_0\|_{\boldsymbol{L}^2(\Omega)}$, we have:

$$\frac{1}{2}\|\widetilde{\boldsymbol{v}}_h(t)\|_{0,2,\Omega}^2 + \nu\int_0^t \left\|D\big(\widetilde{\boldsymbol{v}}_h(s)\big)\right\|_{0,2,\Omega}^2 ds \quad + \int_0^t \langle G\big(\widetilde{\boldsymbol{v}}_h(s)\big), \widetilde{\boldsymbol{v}}_h(s)\rangle\, ds$$
$$\leq \frac{1}{2}\|\boldsymbol{v}_0\|_{0,2,\Omega}^2 + \int_0^{t_k(t)} \langle \boldsymbol{f}(s), \widetilde{\boldsymbol{v}}_h(s)\rangle\, ds \;\leq \frac{1}{2}\|\boldsymbol{v}_0\|_{0,2,\Omega}^2 \tag{3.3}$$
$$+ \int_0^t \langle \boldsymbol{f}(s), \widetilde{\boldsymbol{v}}_h(s)\rangle\, ds + C\sqrt{\Delta t}\ .$$

And yet, according to the work of Brézis (1983), we deduce

$$\frac{1}{2}\|\boldsymbol{v}(t)\|_{0,2,\Omega}^2 + \nu\int_0^t \left\|D\big(\boldsymbol{v}(s)\big)\right\|_{0,2,\Omega}^2 ds \quad + \int_0^t \langle G\big(\boldsymbol{v}(t)\big), \boldsymbol{v}(t)\rangle\, ds$$
$$\leq \lim_{(h,\Delta t)\longrightarrow 0}\left(\frac{1}{2}\|\widetilde{\boldsymbol{v}}_h(t)\|_{0,2,\Omega}^2 + \nu\int_0^t \left\|D\big(\widetilde{\boldsymbol{v}}_h(t)\big)\right\|_{0,2,\Omega}^2 ds \right. \tag{3.4}$$
$$\left. + \int_0^t \langle G\big(\widetilde{\boldsymbol{v}}_h(t)\big), \widetilde{\boldsymbol{v}}_h(t)\rangle\, ds\right).$$

which, combined to (3.3), proves (3.4) ■

In this proof the subgrid dissipation energy term,

$$E_S(\widetilde{\boldsymbol{v}}_h) = \boldsymbol{C}_S^2\int_0^T \sum_{K\in\mathcal{J}_h} h_K^2\left\|D\big(\widetilde{\boldsymbol{v}}_h(t)\big)\right\|_{0,3,K}^3 dt$$

has been treated only using that it is positive. And, assume that the family of grids $(\mathcal{J}_h)_{h>0}$ is regular. Then, the problem (2.1) admits a unique solution. Moreover, this solution satisfies the following estimates

$$\|\boldsymbol{v}_h\|_{L^\infty(\boldsymbol{L}^2)} + \sqrt{\nu}\,\|\boldsymbol{v}_h\|_{L^2(\boldsymbol{H}^1)} + h_{min}\|D(\boldsymbol{v}_h)\|^{3/2}_{L^3(\boldsymbol{L}^3)},$$

it is uniformly bounded with respect to h and Δt. However, the stability $L^\infty(\boldsymbol{L}^2)$ and $L^2(\boldsymbol{H}^1)$ estimates, combined with inverse inequalities, are not sufficient to prove that $E_S(\boldsymbol{v}_h)$ asymptotically vanishes.

As

$$\begin{cases} \left\|D\left(\tilde{\boldsymbol{v}}_h(t)\right)\right\|_{0,3K} \leq C h_K^{-1-d/6}\|\tilde{\boldsymbol{v}}_h(t)\|_{0,2,K}, \\ \left\|D\left(\tilde{\boldsymbol{v}}_h(t)\right)\right\|_{0,3K} \leq C h_K^{-d/6}\|\tilde{\boldsymbol{v}}_h(t)\|_{0,2,K}, \end{cases} \tag{3.5}$$

we deduce

$$E_S(\tilde{v}_h) \leq \boldsymbol{C}_S^2 h_{min}^{1-d/2}\|\tilde{v}_h\|_{L^\infty(\boldsymbol{L}^2)}\|\tilde{v}_h\|_{L^2(\boldsymbol{H}^1)}.$$

An eddy viscosity of order h^α with $\alpha > 1 + {}^{d}/_{2}$ instead of $\alpha = 2$ would ensure that $E_S(\tilde{v}_h)$ asymptotically vanishes.

4. Approximation of weak solution of the LES–Smagorinsky model

The present analysis can be extended to the Euler method approximation of the LES–Smagorinsky model, already discussed in previous works by Santos & Sales (2023) and Santos & Silva (2023). The Eq. (2.1) is changed into a similar one, with the only replacement of the form $\boldsymbol{C}$ by

$$\partial(\boldsymbol{v};\boldsymbol{w}) = \boldsymbol{C}_S^2\delta^2(|D\boldsymbol{v}|\,D\boldsymbol{v}, D\boldsymbol{w})_\Omega :$$

Obtain $\boldsymbol{v}_h^{n+1} \in \boldsymbol{W}_h$, $p_h^{n+1} \in \boldsymbol{M}_h$ such that for all $\boldsymbol{w}_h \in \boldsymbol{W}_h, q_h \in \boldsymbol{M}_h$,

$$\begin{cases} \left(\dfrac{\boldsymbol{v}_h^{n+1} - \boldsymbol{v}_h^n}{\Delta t}, \boldsymbol{w}_h\right)_\Omega + b(\boldsymbol{v}_h^n; \boldsymbol{v}_h^{n+1}, \boldsymbol{w}_h) + a(\boldsymbol{v}_h^{n+1}, \boldsymbol{w}_h) + \partial(\boldsymbol{v}_h^{n+1}, \boldsymbol{w}_h) + \langle G(\boldsymbol{v}_h^{n+1}), \boldsymbol{w}_h\rangle \\ \qquad -(p_h^{n+1}, \nabla\cdot\boldsymbol{w}_h)_\Omega = \langle \boldsymbol{f}^{n+1}, \boldsymbol{w}_h\rangle, \\ \qquad (\nabla\cdot\boldsymbol{v}_h^{n+1}, q_h)_\Omega = 0, \end{cases} \tag{3.6}$$

And yet, estimating $\boldsymbol{v}_h^{n+1} \in \boldsymbol{W}_h$, we get:

$$\|\boldsymbol{v}_h^{n+1}\|_{0,2,\Omega}^2 + \sum_{n=0}^{k} \|\boldsymbol{v}_h^{n+1} - \boldsymbol{v}_h^{n}\|_{0,2,\Omega}^2 + \nu\, \Delta t \sum_{n=0}^{k} \left\|D\left(\boldsymbol{v}_h^{k+1}\right)\right\|_{0,2,\Omega}^2 + 2\, \Delta t \sum_{n=0}^{k} \langle G(\boldsymbol{v}_h^{n+1}), \boldsymbol{v}_h^{n+1}\rangle + 2\, \boldsymbol{C}_S^2 h_{min}^2\, \Delta t \sum_{n=0}^{k} \|D(\boldsymbol{v}_h^{n+1})\|_{0,3,\Omega}^3 \leq \|\boldsymbol{v}_{0h}\|_{0,2,\Omega}^2 + 4\, \Delta t\, \nu^{-1} \sum_{n=0}^{k} \|\boldsymbol{f}^{n+1}\|_{W_{D(\Omega)'}}^2, \tag{3.7}$$

yields the additional stability of this approximation in $L^3(\boldsymbol{W}^{1,3})$, as it holds with h_{min} changed into δ. This allows to prove the weak convergence of the sequence $(\boldsymbol{v}_h)_{h>0}$ to a weak solution of problem (2.1) in $L^3(\boldsymbol{W}^{1,3})$. This is the well-known regularity of the weak solution of the LES–Smagorinsky model. A complete analysis of the approximation of LES models to weak solutions can be found in John (2006) and Parés (1992).

In the work proposed by the authors Siddiqua & Xie (2023), order, while in our work, the temporal advance is to the second order, preserving the kinetic energy model, without the need for other terms that can "extrapolate" the numerical methodology, preserving the physical quantities of interest, such as energy itself kinetic, viscous dissipation without numerical diffusion. However, the aforementioned authors presented a solution to the classical Smagorinsky model, proposing an approximation for an average velocity (resolved). That, as it is a turbulent viscosity model, it cannot represent the energy flow from unresolved fluctuations to the average (resolved) velocity. However, they carry out a complete numerical analysis, presenting two algorithms for their approximation, proving the effectiveness of their methodology, something that will be proven in future work regarding the implementation of the mathematical analysis used in this work.

5. Conclusion

In this section, we delve into the numerical approximation of the non-stationary Navier-Stokes equations within a turbulent regime using the Smagorinsky Model (SM). Similar to the stationary case, we conceive of this model as inherently discrete. Our choice for a semi-implicit time discretization involves utilizing the Euler method, providing a temporal framework for the model. We conducted thorough stability analyses, proposing a well-formulated approach applicable to all flow regimes. Additionally, we explored the asymptotic balance of energy during dynamic fluid movements. Building on this investigation, we identify potential avenues for future research. Firstly, exploring alternative temporal discretization methods holds promise for assessing their impact on both model stability and accuracy. Moreover, extending this methodology to incorporate more intricate boundary conditions or irregular geometries could enhance the model's versatility in addressing real-world scenarios.

Another promising direction involves considering variations of the Smagorinsky Model tailored to address specific nuances within turbulent regimes. Investigating advanced numerical resolution methods and exploring the integration of machine learning techniques to optimize both computational efficiency and model accuracy represent valuable research paths. Furthermore, the practical implementation of the model in simulations of real-world cases, coupled with a thorough comparison of results against experimental data, could yield additional insights into the validity and applicability of the Smagorinsky Model within practical contexts. These proposed avenues for future research aim to advance our comprehension of the model, augmenting its utility and robustness in practical applications of fluid dynamics.

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