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A brief functional investigation via Sobolev spaces to analyze the behavior of Oseen flow in external domains

Uma breve investigação funcional via espaços de Sobolev para analise do comportamento do escoamento de Oseen em domínios externos

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Resumo
Este trabalho serve como uma exploração introdutória do escoamento de Oseen, uma estrutura matemática e física para modelagem de dinâmica de fluidos, com foco específico em substâncias viscosas como líquidos ou gases. O teorema apresentado e seus lemas associados são uma extensão da teoria fundamental do fluxo potencial, que é convencionalmente aplicada a fluidos invíscidos. O objetivo principal na investigação do escoamento de Oseen dentro de domínios externos é elucidar a influência dos efeitos viscosos nos padrões do escoamento além das estruturas sólidas. Como resultado, provamos a existência de soluções generalizadas quando Ω é um domínio tridimensional (3D) no problema de Stokes.
Palavras-chave: Escoamento de Oseen. Domínio 3D. Domínio Externo. Fluido Invíscido.

Abstract
This work serves as an introductory exploration of Oseen flow, a mathematical and physical framework for modeling fluid dynamics, with a specific focus on viscous substances like liquids or gases. The presented theorem and its associated lemmas are an extension of the foundational theory of potential flow, which is conventionally applied to inviscid fluids. The primary goal in investigating Oseen flow within external domains is to elucidate the influence of viscous effects on flow patterns beyond solid structures. As a result, we prove the existence of generalized solutions when Ω is a three-dimensional (3D) domain in Stokes problem.
Keywords: Oseen Flow. 3D Domain. External Domain. Inviscid Fluid.

Important annotations, comments and mathematical formulations

We will consider, in this work, a viscous fluid with constant density, called viscous liquid ($\mathcal{L}$), moving within a predetermined region Ω of three-dimensional space $\mathbb{R}^3$. We will assume that the general movement of the viscous liquid $\mathcal{L}$, in relation to an inertial frame of reference, can be characterized by the following system of equations:

$$\begin{aligned}&\rho\left(\frac{\partial v}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right)=\mu\Delta v-\nabla\pi-\rho\mathbf{f}(x,t),\\&\nabla\cdot\boldsymbol{v}=0,\end{aligned}\qquad(0.1)$$

where t is the time, $x=(x_1,x_2,x_3)$ is a point of Ω, ρ is the constant density of $\mathcal{L}$, $\boldsymbol{v}=v(x,t)=\big(v_1(x,t),v_2(x,t),v_3(x,t)\big)$ and $\pi=\pi(x,t)$ are the Eulerian velocity and pressure fields, respectively, and the positive constant μ is the shear viscosity coefficient. Moreover,

$$\boldsymbol{v}\cdot\nabla\boldsymbol{v}\equiv\sum_{i=1}^{3}v_i\frac{\partial v}{\partial x_i}$$

is the convective term, and

$$\Delta\equiv\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}$$

is the Laplace operator, while

$$\nabla\equiv\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)$$

is the gradient operator, and

$$\nabla\cdot\boldsymbol{v}\equiv\sum_{i=1}^{3}\frac{\partial v_i}{\partial x_i}$$

is the divergence of $\boldsymbol{v}$**.** Finally, $\mathbf{f}$ is the external force per unit mass (body force) acting on $\mathcal{L}$. The Eq. (0.1) expresses the balance of linear momentum (Newton's law), which in turn guarantees that the velocity field is solenoidal, coupled to the mass conservation equation (incompressibility condition). Note that, unlike the compressible scheme, the pressure π in this context is not considered a thermodynamic variable. Instead, it serves as the 'reaction force' necessary to maintain the integrity of the material volume of the viscous liquid $\mathcal{L}$.

Throughout this study, we will focus on stationary motions, where the system of Eq. (0.1), with $\partial v/\partial t\equiv 0$ and $\mathbf{f}=\mathbf{f}(x)$**,** takes the form

$$\begin{aligned}&\nu\Delta\boldsymbol{v}=\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\mathbf{f},\\&\nabla\cdot\boldsymbol{v}=0,\end{aligned}\qquad(0.2)$$

where $p=\frac{\pi}{\rho}$, and $\nu=\frac{\mu}{\rho}$ is the kinematic viscosity coefficient. We shall continue to call p "the pressure" (or "the pressure field") of the liquid $\mathcal{L}$. To conduct our investigation, it is essential to supplement Eq. (0.2) with suitable physical conditions. These conditions depend on the specific physics of the problem at hand, which, in a broader sense, can be categorized based on the nature of the region Ω where fluid flow occurs. To resolve this, we will differentiate the following cases: (*i*) Ω is a bounded, (*ii*) Ω is the complement of a bounded region (i.e., Ω is an exterior region). In both circumstances Ω

has a bounded boundary. Typically, in confined areas, the driving force behind fluid movement stems from either the motion of a portion of the boundary, as seen in the flow between two rotating, concentric spheres, or from the introduction and extraction of liquid through the permeable section of the boundary. This scenario is exemplified in flows occurring within a region containing a finite number of "sources" and "sinks." It is worth noting that in particular, condition $\boldsymbol{v}(y) = \boldsymbol{v}_*(y), y \in \partial\Omega$, often referred to as a no-slip boundary condition, requires that the particles of the liquid "adhere" to the boundary $\partial\Omega$ in the case that $\partial\Omega$ is motionless, rigid, and impermeable ($\boldsymbol{v}_* \equiv 0$).

Because of the solenoidality condition, the Eq. (0.2), and in view of Gauss theorem, it turns out that the field $\boldsymbol{v}_*$ must satisfy the compatibility condition:

$$\int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n}\, dS = 0 \tag{0.3}$$

with $\boldsymbol{n}$ unit outer normal to $\partial\Omega$. In other words, in mathematical and physical terms, the expression can be interpreted as a condition in which the flow through the surface is zero, indicating a condition of "non-penetration" or, better known as mass balance in a given fluid domain.

1. Introduction

The Oseen approximation is developed to elucidate the dynamics of fluids around rigid bodies, labeled in this work as $\mathcal{B}$, experiencing constant and 'small enough' purely translational motion within a fluid, where its dynamics are described by the governing Navier equations. Stokes. However, assuming this type of movement, around $\mathcal{B}$, this may in turn impose limitations in several practical physical scenarios. These limitations can be observed on macroscopic and microscopic scales, where in $\mathcal{B}$ not only translational but also rotational movement is allowed. Other examples of these situations may include the alignment of rigid bodies (bodies in tandem) in the flow of a viscous fluid and the autonomous movement of microorganisms in a viscous medium (which will not be considered in this work); the work of Galdi (2002) offers a comprehensive exploration of these and related topics. Furthermore, as Sobolev's functional space will be mentioned, for the study in question, we refer the reader to the works of Santos & Sales (2023) and Santos & Silva (2023).

In this context, to adequately describe situations in which body $\mathcal{B}$ moves through a generic, but "small" and rigid motion, a more general approximation is necessary, which we introduce via Eq. (1), which we will call the generalized Oseen approximation. For the reader's better understanding, we reproduce the relevant equations here:

$$\begin{cases} \nu\Delta\boldsymbol{v} + \boldsymbol{v}_0 \cdot \nabla\boldsymbol{v} + \omega \times \boldsymbol{x} \cdot \nabla\boldsymbol{v} - \omega \times \boldsymbol{v} = \nabla p + \mathbf{f} & in \quad \Omega, \\ \nabla \cdot \boldsymbol{v} = 0 & in \quad \Omega, \\ \boldsymbol{v} = \boldsymbol{v}_* & at \quad \partial\Omega, \end{cases} \tag{1}$$

together with the condition at infinity

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = 0. \tag{2}$$

We remind that in Eq. (1), $\boldsymbol{v}_0$ and ω are provided as constant vectors, symbolizing the translational and angular velocity, correspondingly, in the rigid motion of $\mathcal{B}$. We will presume that Ω is an exterior domain of $\mathbb{R}^3$, and furthermore, we emphasize the importance of reviewing pertinent observations and bibliographies (like this, Ladyzhenskaia (1959) and Solonnikov (1996)) related to the two-dimensional case that will be discussed in this work.

To articulate problems and outcomes more effectively, it is advantageous to rephrase Eq. (1) and Eq. (2) in a suitable dimensionless form. To this end, we assume, without loss, that ω is directed along the positive $x_1 -$ axis, that is, $\omega = \omega e_1$, while $\boldsymbol{v}_0 = \boldsymbol{v}_0 e$, $\boldsymbol{v}_0 \geq 0$ (Of course, we suppose $\omega \neq 0$). Furthermore, we normalize the length by $d = \delta(\Omega^c)$ (which refers to the Dirac distribution associated with the complement Ω^c of a set Ω, and when applying it to testable functions, it provides a weight on points outside the set Ω. This approach is commonly used in Sobolev spaces and distribution theory to treat lumped functions in specific sets in a generalized way.), and the velocity with $\boldsymbol{v}_0$, if $\boldsymbol{v}_0 \neq 0$, and ωd otherwise. Therefore, introducing the dimensionless numbers

$$\begin{aligned} \mathcal{R}' &= \frac{v_0 d}{\nu} \; (Reynolds\ number), \\ \mathcal{T} &= \frac{\omega d^2}{\nu} \; (Taylor\ number), \end{aligned} \tag{3}$$

the system Eq. (1) assumes the following form

$$\begin{cases} \Delta \boldsymbol{v} + \mathcal{R}' e \cdot \nabla \boldsymbol{v} + \mathcal{T}(e_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - e_1 \times \boldsymbol{v}) = \nabla p + \mathbf{f} & in \quad \Omega, \\ \nabla \cdot \boldsymbol{v} = 0 & in \quad \Omega, \\ \boldsymbol{v} = \boldsymbol{v}_* & at \quad \partial\Omega, \end{cases} \tag{4}$$

where now $\boldsymbol{v}$, $\boldsymbol{v}_*, p$, and $\mathbf{f}$ are nondimensional quantities. If $\Omega \equiv \mathbb{R}^3$ the above choice of d is no longer possible, but we can still give a meaning to Eq. (4), which is what we shall do hereinafter.

At this point we observe that, in general, ω and $\boldsymbol{v}_0$, that is, e_1 and e, have different directions. However, by shifting the coordinate system by a constant quantity, we can always reduce the original equations to new ones where $e = e_1$. This change of coordinates, known as Mozzi–Chasles transformation (see more in Caparrini (2003) and Ceccarelli (2007)), reads as follows:

$$\boldsymbol{x}^* = \boldsymbol{x} - \lambda \boldsymbol{e_1} \times e, \lambda := \frac{\mathcal{R}'}{T} \equiv \frac{v_0}{\omega d}. \tag{5}$$

Thus, defining

$$\begin{aligned} &\Omega^* = \{\boldsymbol{x}^* \in \mathbb{R}^3 : \boldsymbol{x}^* = \boldsymbol{x} - \lambda \boldsymbol{e_1} \times e, for\ some\ \boldsymbol{x} \in \Omega\}, \\ &\boldsymbol{v}^*(\boldsymbol{x}^*) = \boldsymbol{v}(\boldsymbol{x}^* + \lambda \boldsymbol{e_1} \times e),\ p^*(x^*) = p(\boldsymbol{x}^* + \lambda \boldsymbol{e_1} \times e), \\ &\mathbf{f}^*(\boldsymbol{x}^*) = \mathbf{f}(\boldsymbol{x}^* + \lambda \boldsymbol{e_1} \times e), \\ &\mathcal{R} = \mathcal{R}' e \cdot e_1, \end{aligned} \tag{6}$$

the Eq. (4) becomes (with stars omitted)

$$\begin{cases} \Delta \boldsymbol{v} + \mathcal{R} \dfrac{\partial v}{\partial x_1} + \mathcal{T}(e_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v} - e_1 \times \boldsymbol{v}) = \nabla p + \mathbf{f} & in \quad \Omega, \\ \nabla \cdot \boldsymbol{v} = 0 & in \quad \Omega, \\ \boldsymbol{v} = \boldsymbol{v}_* & at \quad \partial\Omega, \end{cases} \tag{7}$$

Throughout this section, we will therefore focus on solving equations Eqs. (7) and (2). We wish to highlight the important feature occurring in Equation (7), as follows. Building upon the prior analysis, we can infer that the wake characteristics of the velocity field $\boldsymbol{v}$ at considerable distances are attributed to the term $\mathcal{R}\frac{\partial v}{\partial x_1}$. Currently, this expression equals zero whenever the "effective" Reynolds number $\mathcal{R}$ becomes negligible. Considering the Mozzi-Chasles transformation, this occurs not only as intuitively anticipated, that is, with $\boldsymbol{v} = 0$, but, more generally, when $v_0 \cdot \omega = 0$, namely, when the translational velocity of the body is perpendicular to its angular velocity. In fact, as we shall prove later on, the formation of a “wake” is possible, in a suitable sense, if and only if, $v_0 \cdot \omega \neq 0$.

The study of the mathematical properties of the solutions of Eq. (7), (2) is, in principle, much more challenging than the analogous study carried out for the Oseen problem Eq. (1), (2), the main reason being the presence, in Eq. (7), of the term $e_1 \times \boldsymbol{x} \cdot \nabla \boldsymbol{v}$, whose coefficient becomes unbounded as $|x| \to \infty$. An important consequence of this fact is that Eq. (7) can in no way be seen as a perturbation of Eq. (1), even for “small” $\mathcal{T}$ (i.e., “small” angular velocities). Despite this challenge, it is feasible to demonstrate, quite readily, the existence of at least one comprehensive solution to Eqs. (7) and (2) while also establishing that this solution remains smooth given equally smooth input data. Similar to the Oseen approximation scenario, the generalized solution is formulated employing the Galerkin method, utilizing a suitable basis and a pertinent a priori estimate of the Dirichlet norm of $\boldsymbol{v}$. This estimation can be established due to the fact that (as the reader will promptly observe)

$$\int_{\Omega} (e_1 \times \boldsymbol{x} \cdot \nabla\varphi \cdot \varphi - e_1 \times \varphi \cdot \varphi)\, dS = 0, \forall\, \varphi \in \mathcal{D}(\Omega). \tag{8}$$

Considering applications to the nonlinear problem, similar to the Oseen approximation, the subsequent study in this particular case revolves around examining the behavior of generalized solutions at significant distances. This question arises naturally when investigating the uniqueness of generalized solutions, a distinct concept is presented in Theorem 1 and two important Lemmas for Oseen approximations, respectively.

Theorem 1. Let Ω be a locally Lipschitz exterior domain of $\mathbb{R}^3$. Given $\mathbf{f} \in D_0^{-1,2}(\Omega)$, $\boldsymbol{v}_* \in W^{1/2,2}(\partial\Omega)$, there exists one and only one generalized solution to the Stokes problem, given by

$$\text{in } \Omega \begin{cases} \Delta \boldsymbol{v} = \nabla p + \mathbf{f}, \\ \nabla \cdot \boldsymbol{v} = 0, \end{cases} \tag{9}$$

$$\boldsymbol{v} = \boldsymbol{v}_* \text{ at } \partial\Omega.$$

where $\mathbf{f}$, $\boldsymbol{v}_*$ are prescribed fields and where, as usual, we have formally set the coefficient of kinematic viscosity to be one. Of course, since Ω is unbounded, we have to assign also the velocity at infinity, which we do as follows

$$\lim_{|x|\to\infty} \boldsymbol{v}(x) = 0.$$

This solution satisfies for all $\mathcal{R} > \delta(\Omega^c)$ the following estimate

$$\|\boldsymbol{v}\|_{2,\Omega_R} + |\boldsymbol{v}|_{1,2} + \|p\|_2 \leq c\{|\mathbf{f}|_{-1,2} + \|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}\}, \tag{10}$$

where p is the pressure field associated to $\boldsymbol{v}$ and $c = c(\Omega, \mathrm{R}),\ c \to \infty$ as $\mathrm{R} \to \infty$.

Furthermore,

$$\int_{S^2} |\boldsymbol{v}(x)|\, dS = o\left(\frac{1}{\sqrt{|x|}}\right), |x| \to \infty. \tag{11}$$

Proof. The proof of existence and uniqueness goes exactly as in ***Theorem 1***, provided we make a suitable extension of $\boldsymbol{v}_*$. In this respect, it is worth noticing that it is not required that the flux of $\boldsymbol{v}_*$ on $\partial\Omega$ be zero. Set

$$\Phi = \int_{\partial\Omega} \boldsymbol{v}_* \cdot \boldsymbol{n}\, dS, \qquad \boldsymbol{\sigma}(x) = -\Phi\nabla\mathcal{E}(x), \tag{12}$$

where $\mathcal{E}$ is the fundamental solution to the Laplace equation and with the origin of coordinates taken in Ω^c. Recall that $\boldsymbol{n}$ is the unit outer normal to $\partial\Omega$. Clearly,

$$\Delta\boldsymbol{\sigma} = 0 \ in\ \Omega,$$

$$\int_{\partial\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{n}\, dS = \Phi.$$

Putting $\boldsymbol{w}_* = \boldsymbol{v}_* - \boldsymbol{\sigma}$, it follows that

$$\boldsymbol{\sigma} = \frac{\Phi}{4\pi}\nabla(|x|^{-1}),$$

$$\int_{\partial\Omega} \boldsymbol{w}_* \cdot \boldsymbol{n}\, dS = 0,$$

and to construct a solenoidal field $\boldsymbol{V}_1 \in \boldsymbol{W}^{1,2}(\Omega)$, vanishing outside Ω_p, for some $\rho > \delta(\Omega^c)$, that equals $\boldsymbol{w}_*$ on $\partial\Omega$ and, moreover,

$$\|\boldsymbol{V}_1\|_{1,2,\Omega_\rho} \leq c_1\|\boldsymbol{w}_*\|_{1/2,2(\partial\Omega)} \tag{13}$$

with $c = c(\Omega_\rho)$. On the other hand, we have, clearly,

$$\|\boldsymbol{w}_*\|_{1/2,2(\partial\Omega)} \leq c_2\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}, \tag{14}$$

so that Eq. (14) implies

$$\|\boldsymbol{V}_1\|_{1,2,\Omega_\rho} \leq c_3\|\boldsymbol{v}_*\|_{1/2,2(\partial\Omega)}, \tag{15}$$

with $c_3 = c_3(\Omega, \rho)$. A generalized solution to the exterior problem is then sought in the form

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{V}_1 + \boldsymbol{\sigma},$$

where $\boldsymbol{w} \in \mathcal{D}_0^{1,2}(\Omega)$ solves

$$(\nabla \boldsymbol{w}, \nabla \varphi) = -[\mathbf{f}, \varphi] - (\nabla \boldsymbol{V}, \nabla \varphi),$$

with

$$\boldsymbol{V} = \boldsymbol{V}_1 + \boldsymbol{\sigma}.$$

The existence, uniqueness, and estimate of Eq. (10) are proved along the same lines of *Lemma 1*, shown below.

Lemma 1. Let Ω be an arbitrary domain of $\mathbb{R}^n$, $n \geq 2$, and let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,q}(\Omega'), 1 < q < \infty$, for any bounded domain $\Omega'$with $\overline{\Omega'} \subset \Omega$. A vector field $\boldsymbol{v} \in \boldsymbol{W}_{loc}^{1,q}(\Omega)$ satisfies $(\nabla \boldsymbol{v}, \nabla \psi) = -\langle \boldsymbol{f}, \psi \rangle + (p, \nabla \cdot \psi)$ for all $\varphi \in \mathcal{D}(\Omega)$ if and only if there exists a pressure field $p \in L_{loc}^q(\Omega)$ such that $(\nabla \boldsymbol{v}, \nabla \psi) = -\langle \boldsymbol{f}, \psi \rangle + (p, \nabla \cdot \psi)$ holds for every $\psi \in C_0^\infty(\Omega)$. If, moreover, Ω is bounded and satisfies the cone condition and $\boldsymbol{f} \in D_0^{-1,q}(\Omega), \boldsymbol{v} \in D^{1,q}(\Omega)$ then $p \in L^q(\Omega)$.

Finally, if we normalize p by the condition

$$\int_\Omega p = 0, \tag{L.1}$$

the following estimate holds:

$$\|p\|_q \leq c\big(\|\boldsymbol{f}\|_{-1,q} + |\boldsymbol{v}|_{1,q}\big). \tag{L.2}$$

Proof. Let us consider the functional

$$\mathcal{F}(\psi) \equiv (\nabla \boldsymbol{v}, \nabla \psi) + \langle \mathbf{f}, \psi \rangle$$

for $\psi \in D_0^{1,q'}(\Omega')$. By assumption, $\mathcal{F}$ is bounded in $D_0^{1,q'}(\Omega')$ and is identically zero in $\mathcal{D}(\Omega)$ and, therefore, by continuity, in $\mathcal{D}_0^{1,q'}(\Omega')$. If Ω is arbitrary (in particular, has no regularity), we deduce the existence of $p \in L^q(\Omega)$ for all $\psi \in C_0^\infty(\Omega)$. If Ω is bounded and satisfies the cone condition, by assumption exists a uniquely determined $p' \in L^q(\Omega)$ with

$$\int_\Omega p' = 0,$$

such that

$$\mathcal{F}(\psi) = (p', \nabla \cdot \psi), \tag{L.3}$$

for all $\psi \in D_0^{1,q'}(\Omega)$. As $(\nabla \boldsymbol{v}, \nabla \psi) = -\langle \boldsymbol{f}, \psi \rangle + (p, \nabla \cdot \psi)$ and $\mathcal{F}(\psi) = (p', \nabla \cdot \psi)$, we find, in particular,

$$(p - p', \nabla \cdot \psi) = 0, \forall\, \psi \in C_0^\infty(\Omega),$$

Implying $p = p' + constant.$, and so, if we normalize p by (L.1), we may take $p = p'$. Considering the problem

$$\begin{aligned} &\nabla \cdot \psi = |p|^{q-2}p - |\Omega|^{-} \int_\Omega |p|^{q-2}p \equiv g \\ &\psi \in W_0^{1,q'}(\Omega') \\ &\|\psi\|_{1,q'} \leq c_1 \|p\|_q^{q-1}, \end{aligned} \tag{L.4}$$

with Ω bounded and satisfying the cone condition. Since

$$\int_\Omega g = 0, \qquad g \in L^{q'}(\Omega), \qquad \|g\|_{q'} \leq c_2 \|p\|_q^{q-1},$$

we deduce the existence of ψ solving (L.4). If we replace such a ψ into (L.3) and use (L.1) together with the Hölder inequality, we obtain (L.2). The proof is therefore completed. ■

And now, provided we use *Lemma 1*, and note that, since $\Delta \boldsymbol{\sigma} = 0$ in Ω, we have:

$$\int_\Omega \nabla \boldsymbol{\sigma} : \nabla \varphi = 0,$$

for any $\varphi \in \mathcal{D}_0^{1,2}(\Omega)$. To show estimate Eq. (11) we notice that for $|x|$ sufficiently large

$$\int_{S^2} |\boldsymbol{v}(x)|\, dS \leq c_4 \int_{S^2} (|\boldsymbol{w}(x)| + |\Phi||\nabla \mathcal{E}(x)|)\, dS = c_4 \int_{S^2} |\boldsymbol{w}(x)|\, dS + \mathcal{O}\left(\frac{1}{|x|^2}\right),$$

and, since $\boldsymbol{w} \in \mathcal{D}_0^{1,2}(\Omega)$, it follows

$$\int_{S^2} |\boldsymbol{w}(x)|\, dS = o\left(\frac{1}{\sqrt{|x|}}\right),$$

which furnishes the Eq. (11). The proof of the *Theorem 1* is then completed. ■

Lemma 2. Let Ω be an arbitrary domain of $\mathbb{R}^n$, $n \geq 2$, then, there exists a denumerable set of functions $\{\varphi_k\}$ whose linear hull is dense in $\mathcal{D}_0^{1,2}(\Omega)$ and has the following properties:

(i) $\varphi_k \in \mathcal{D}(\Omega), \forall k \in \mathbb{N}$;
(ii) $(\nabla\varphi_k, \nabla\varphi_j) = \delta_{kj}\ or\ (\varphi_k, \varphi_j) = \delta_{kj}, \forall k, j \in \mathbb{N}$;
(iii) $Given\ \varphi \in \mathcal{D}(\Omega), and\ k \in \mathbb{N}$, for any $\varepsilon > 0$ there exist $m = m(\varepsilon) \in \mathbb{N}$, and $\gamma_1, \dots, \gamma_m \in \mathbb{R}$, such that

$$\left\| \nabla\varphi - \sum_{i=1}^{m} \gamma_i \nabla\varphi_i \right\|_s + \left\| (|x| + 1)^{k/s} \left(\varphi - \sum_{i=1}^{m} \gamma_i \nabla\varphi_i \right) \right\|_s < \varepsilon,$$

for all $s \geq 2$, where $(|x| + 1)^{k/s} = \rho$.

Proof. Let $H_{0,\rho}^{\ell}(\Omega)$, with $\ell > n/2 + 1$, be completion of $\mathcal{D}(\Omega)$ in the norm

$$\|\varphi\|_{\ell,2,\rho} \equiv \|\rho\varphi\|_2 + \|\varphi\|_{\ell,2} \,.$$

Clearly, $H_{0,\rho}^{\ell}(\Omega)$ is a subspace of $\boldsymbol{W}^{\ell,2}(\Omega)$. Moreover, it is also isomorphic to a closed subspace of $[L^2(\Omega)]^N$, for suitable $N = N(\ell, n)$, via the map

$$\varphi \in H_{0,\rho}^{\ell}(\Omega) \to \left(\rho\varphi_1, \dots, \rho\varphi_n; (D^\alpha\varphi_1)_{1\leq[\alpha]\leq\ell}; \dots; (D^\alpha\varphi_n)_{1\leq[\alpha]\leq\ell}\right) \in [L^2(\Omega)]^N \,.$$

Thus, in particular, $H_{0,\rho}^{\ell}(\Omega)$ is separable, and so is its subset $\mathcal{D}(\Omega)$. As a consequence, there exists a basis in $H_{0,\rho}^{\ell}(\Omega)$ of functions from $\mathcal{D}(\Omega)$, which we will denote by $\{\psi_k\}$. Since $H_{0,\rho}^{\ell}(\Omega) \hookrightarrow \mathcal{D}_0^{1,2}(\Omega)$, the linear hull of $\{\psi_k\}$ must be dense in $\mathcal{D}_0^{1,2}(\Omega)$ as well. Take $\varphi \in \mathcal{D}(\Omega)$ and fix $\varepsilon > 0$; there exist $N = N(\varepsilon) \in \mathbb{N}$, and $\alpha_1, \dots, \alpha_N \in \mathbb{R}$ such that

$$\left\| \varphi - \sum_{i=1}^{N} \alpha_i \psi_i \right\|_{\ell,2,\rho} < \varepsilon \,,$$

it follows that

$$\left\| \varphi - \sum_{i=1}^{N} \alpha_i \psi_i \right\|_{C^1} < c\varepsilon \,,$$

with $c = c(\Omega, n, \ell)$. We may orthonormalize $\{\psi_k\}$ in $\mathcal{D}_0^{1,2}(\Omega)$ by the Schmidt procedure, to obtain another denumerable set $\{\varphi_k\}$ whose linear hull is still dense in $\mathcal{D}_0^{1,2}(\Omega)$. Since every φ_r is a linear combination of $\psi_1, \dots, \psi_r$ and, conversely, every ψ_r is a linear combination of $\varphi_1, \dots, \varphi_r$, it is easy to check that the system $\{\varphi_k\}$ satisfies all the statements in the lemma which is thus completely proved. ■

5. Conclusion

Thus, we conclude, based on the theorem presented and its results extended as lemmas, that (i) the conditions under which a solution to a differential equation involving a vector field $\boldsymbol{v}$ and a pressure field p can be found in a domain Ω are established. Which results in the imposition of restrictions on the functions involved and the domain, providing information about the existence and regularity of the solution; (*ii*) that in the density of a denumerable set of functions $\{\varphi_k\}$ in a Sobolev space $\mathcal{D}_0^{1,2}$ with null boundary condition. The orthogonal properties and density in Sobolev space ensure that this set is an effective basis for representing functions in this space, allowing accurate approximations.

In summary, while the first lemma addresses the existence and regularity of solutions to specific differential equations, the second lemma deals with the density of a set of functions in a Sobolev space, providing an effective basis for representing functions in that space. Both lemmas contribute to theoretical understanding and mathematical analysis in contexts associated with differential equations and functional spaces of type 3D.

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