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Incompressible Schrödinger Flow with Heat-Transfer: An Introduction to the Analysis of Isotropic Fluid Dynamics in Sobolev Spaces for an Immersed Arbitrary Isothermal Geometry

Escoamento Incompressível de Schrödinger com Transferência de Calor: Uma introdução à Análise da Dinâmica de Fluidos Isotrópicos em Espaços de Sobolev para uma Geometria Isotérmica Arbitrária Imersa

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Abstract
This research investigates the intricate interplay of incompressible Schrödinger flow, heat-transfer, and the presence of an immersed isothermal body. The mathematical framework encompasses the Schrödinger equation for incompressible fluids, the heat transfer equation, and introduces a term that represents the thermal influence of an immersed isothermal geometry. Emphasizing the modeling and analysis of isotropic fluid dynamics, the study seeks to unravel the subtle relationship between the principles of quantum mechanics and the classical behavior of fluids. The initial discoveries produce an important theorem that leads the name of the authors, allowing new and valuable insights into the effects of the isothermal body immersed in a fluid medium. As a result, it was found that the term temperature source offers a unique perspective at the intersection of quantum mechanics and fluid dynamics.
Keywords: Incompressible Schrödinger Flow. Isothermal Immersed Geometry. Heat Transfer. Mathematical Modeling.

Resumo
Esta pesquisa investiga a intrincada interação entre o escoamento incompressível de Schrödinger, a transferência de calor e a presença de um corpo isotérmico imerso. A estrutura matemática abrange a equação de Schrödinger para fluidos incompressíveis, a equação de transferência de calor, e introduz um termo que representa a influência térmica de uma geometria isotérmica imersa. Enfatizando a modelagem e análise da dinâmica de fluidos isotrópicas, o estudo busca desvendar a relação sutil entre os princípios da mecânica quântica e o comportamento clássico dos fluidos. As descobertas iniciais produzem um importante teorema que dá origem ao nome dos autores, permitindo novos e valiosos insights sobre os efeitos do corpo isotérmico imerso em um meio fluido. Como resultado, descobriu-se que o termo fonte de temperatura oferece uma perspectiva única na intersecção da mecânica quântica e da dinâmica dos fluidos.

Palavras-chave: Escoamento Incompressível de Schrödinger. Geometria Imersa Isotérmica. Transferência de Calor. Modelagem Matemática.

About mathematical notations

Exploring the nuanced interaction between quantum mechanics and classical fluid dynamics stands as a captivating field of research. In this initial inquiry, we immerse ourselves in the domain of incompressible Schrödinger flow, considering the factors of heat transfer and the existence of an immersed isothermal body. The mathematical formulations utilized lay the groundwork for modeling and scrutinizing the isotropic fluid dynamics within this distinctive context. Notations corresponding to each section are detailed within the body of the text.

1. Introduction

The concept of *Incompressible Schrödinger Flow* (ISF) presents an interesting mathematical structure that draws parallels between the Schrödinger equation in quantum mechanics and the equations that govern incompressible fluids in well-known fluid mechanics. The approach presented in this work seeks to unite two apparently separate domains, investigating the mathematical similarities between quantum phenomena and non-compressible (incompressible) fluid dynamic behavior is challenging. The equations outlined in this study, designed to encapsulate incompressibility with heat-transfer, can be articulated mathematically, illuminating the fundamental connections between quantum mechanics and fluid dynamics. From this point of view, the study not only seeks to unravel the intricate interplay between concepts, but also lays the foundation for a nuanced understanding of the subtle analogies that intertwine the structures of the physical world.

Medeiros & Miranda (2000) in this work, the authors begin a seminar dedicated to Sobolev spaces and their applications in partial differential equations, held at the Brazilian Center for Physics Research in the 1970s. It consisted of instigating young students' interest in this specific aspect of mathematics on a topic of relevance in the study of Functional Analysis. Its results extend into different applications to this day.

Works such as Brezis (2011) offer a coherent, concise, and unified approach to integrating elements from two distinct realms—functional analysis and partial differential equations. The author facilitates a seamless transition between these areas, delving into the intricacies of one-dimensional PDEs, thereby providing a more accessible entry point for beginners.

Turning our attention to Sobolev Spaces, the contributions of Santos & Sales (2023) and Santos & Silva (2023) deserve mention. In a broad sense, these works employ Sobolev functional spaces to scrutinize the asymptotic behavior of turbulent flow in a fluid medium. The mathematical analyses in these works serve as foundational pillars for a more extensive exploration of the regularity of the Navier-Stokes Equations. In this specific context, the authors embark on a crucial step in advancing the Smagorinsky model. Drawing on the frameworks of Banach and Sobolev Spaces, they develop a novel theorem that illuminates the path toward constructing an anisotropic viscosity model. Their dedicated effort initially focuses on presenting a more comprehensive mathematical analysis, thereby fostering a nuanced understanding of the challenges posed by the regularity of the Navier-Stokes equations.

In the work Teschl (2009), the author presents a succinct and autonomous introduction to the mathematical techniques of quantum mechanics, with specific emphasis on their practical application to Schrödinger operators. In the first part of the text, the author delves into the spectral theory of unbounded operators, covering only the fundamental topics essential for subsequent applications, with the spectral theorem taking center stage. In Part 2, Teschi begins the exploration with the free Schrödinger equation, deftly computing the free resolvent and time evolution. Notably, concepts such as position, momentum, and angular momentum are elucidated using algebraic methods.

This particular job by Teschl requires only a robust understanding of advanced calculus and an introductory knowledge of complex analysis. Notably, he makes no assumptions about Lebesgue's functional analysis or integration theory. This approach makes your work accessible and ensures readers can interact with the content effectively.

The second edition of Evans (2022), work on partial differential equations (PDE) is of utmost importance, offering a thorough exploration of contemporary techniques in the theoretical study of Partial Differential Equations (PDE), with specific emphasis on non-linear equations. Recognized for its broad scope and lucid exposition, Evans' (2022) work has been acclaimed by both educators and students of the exact sciences. His distinctive combination of deep knowledge and technical precision makes him a crucial resource for anyone delving into the complexities of PDE. Remarkable for its ability to elucidate fundamental ideas and techniques, this work is highly recommended, establishing itself as an essential reference for various facets of the area.

Thus, the functional analysis presented by Brezis (2011), Evans (2022), Santos & Sales (2023) and Santos & Silva (2023), is crucial to understanding Sobolev spaces, which are frequently used in the analysis of solutions of Partial Differential Equations. Teschl's work on quantum mechanics provides a basis for understanding Schrödinger's equations and their associated operators, and is useful for those exploring quantum phenomena such as incompressible flow with heat-transfer. Evans' work on PDEs offers comprehensive tools to address issues related to heat transfer in a more general context. Although these references do not specifically address incompressible Schrödinger flow with heat-transfer considering an immersed isothermal body, they provide the mathematical and theoretical basis necessary to address problems related to partial differential equations, functional analysis and quantum phenomena. Exploring the intersection of these concepts can be a challenging and innovative task within the field of mathematical physics.

Although we mention some references in an introductory manner, other important ones will be presented and commented succinctly throughout this work. Thus, an important, albeit introductory, step will be taken towards understanding Schrödinger Incompressible Flow with Heat-Transfer.

2. Mathematical analysis of the problem

The incompressible Schrödinger equation draws an analogy between the Schrödinger equation in quantum mechanics and the incompressible fluid equations in fluid mechanics. Initially, the aim is to establish a broad correspondence between these two distinct fields or areas.

The incompressible Schrödinger equation provides an interesting analogy between quantum mechanics and incompressible fluid mechanics. This analogy is established through the mathematical expression:

$$i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\psi+V(\boldsymbol{x},t)\psi=Q(\boldsymbol{x},t), \tag{2.1}$$

where, i is the imaginary unit, $\hbar$ represents the reduced Planck constant, $\frac{\partial\psi}{\partial t}$ is the partial derivative with respect to time, $\nabla^2\psi$ is the spatial Laplacian of the wave function ψ, m is the mass of the particle, and $V(\boldsymbol{x},t)$ is the potential that may depend on both the position x and time t, $Q(\boldsymbol{x},t)$ is the heat source. This formulation, by incorporating elements from the conventional Schrödinger equation of quantum mechanics, provides an intriguing perspective for analyzing complex physical phenomena that share similarities between the two scientific domains.

Thus, the Eq. (2.1), represents the conservation of energy for the particle in the context of quantum mechanics. It describes how the wave function ψ evolves over time under the influence of the potential $V(\boldsymbol{x},t)$ and initial conditions. This specific formulation, with an emphasis on incompressible fluids, can be applied to analyze the quantum behavior of fluid systems, such as, the dynamics of superfluids or Bose-Einstein condensates (just exemplifying). It provides a mathematical tool for investigating quantum phenomena in specific fluid contexts.

2.1 Existence and uniqueness for the incompressible Schrödinger equation with immersed isothermal body and Robin boundary condition

Consider the incompressible Schrödinger equation in a domain $\Omega \subset \mathbb{R}^3$ with an arbitrary two-dimensional isothermal body Γ. The Eq. (2.2), expressed by

$$i\frac{\partial \psi}{\partial t} + \frac{1}{2}\nabla^2\psi + V(\boldsymbol{x},t)\psi = Q(\boldsymbol{x},t), \qquad in\ (0,T)\times\Omega, \tag{2.2}$$

subject to Dirichlet boundary conditions in the parts of $\partial\Omega$ that do not coincide with Γ

$$\psi(\boldsymbol{x},t) = 0, \text{ for all } \boldsymbol{x} \in \ \partial\Omega\backslash\Gamma, t \in (0,T).$$

where, this expression represents the Dirichlet boundary conditions. Each term represents, $\psi(\boldsymbol{x},t)$, the wave function, which is the solution we are looking for to the incompressible Schrödinger equation. It depends on the spatial variables $\boldsymbol{x}$ and temporal variables t. The $\partial\Omega$ represents the domain boundary Ω, which is the three-dimensional (3D) space where we are studying the problem. The Γ represents the region occupied by the immersed isothermal two-dimensional arbitrary body. So, $\partial\Omega\backslash\Gamma$ denotes the part of the border that does not coincide with the region occupied by the body. And, $t \in (0,T)$, indicates that the boundary conditions are applied for all time instants in the interval $0 < t < T$. Therefore, this condition means that the wave function is zero in those parts of the boundary that do not coincide with the region occupied by the arbitrary isothermal body. In other words, this boundary condition specifies the value of the function at the domain boundary. That is, it indicates that the wave function is fixed to zero in these parts of the boundary, to be clear. Now, considering, the Robin boundary condition on the parts of $\partial\Omega$ that coincide with Γ

$$a(\boldsymbol{x},t)\psi + b(\boldsymbol{x},t)\frac{\partial \psi}{\partial n} = g(\boldsymbol{x},t), \text{ for all } \boldsymbol{x} \in \Gamma, t \in (0,T),$$

the previous expression, represents the Robin boundary condition. Some of its terms have already been mentioned, however, the term $a(\boldsymbol{x},t)$ is a known function that describes the influence of the potential at the boundary Γ on the wave function. The term, $b(\boldsymbol{x},t)$, it is also a known function that modulates the normal derivative of the wave function at the Γ boundary. The notation $\frac{\partial \psi}{\partial n}$, represents the normal derivative of the wave function with respect to the variable normal to the boundary. Here, $\frac{\partial}{\partial n}$ is the normal derivative operator; and $g(\boldsymbol{x},t)$, it is a known function that specifies the values on the boundary Γ with respect to time. And still, $\boldsymbol{x} \in \Gamma$ indicates that this boundary condition applies to the boundary region occupied by the arbitrary two-dimensional isothermal body. In summary, the Robin boundary condition establishes a linear relationship between the wave function, its normal derivative at the boundary and known functions $a(\boldsymbol{x},t)$, $b(\boldsymbol{x},t)$ and $g(\boldsymbol{x},t)$. This condition is generally used when one wants to model heat transfer or other physical quantities that depend on the normal flow at the boundary. The specific choice of $a(\boldsymbol{x},t)$, $b(\boldsymbol{x},t)$ and $g(\boldsymbol{x},t)$, will always depend on the physical characteristics of the problem in question.

An important theorem below, called in this work as *"Santos-Sales Theorem (SST)"*, plays a fundamental role in affirming the singular solution of the incompressible Schrödinger equation.

Theorem (Santos-Sales Theorem - SST). *An incompressible Schrödinger equation has a singular solution $\psi \in H_0^1(\Omega)$ within Sobolev space, which acts as the only minimizer of the energy functional E[ψ], under Dirichlet and Robin boundary conditions.*

Before starting the mathematical proof process, it is worth mentioning that the variational formulation to be used is a powerful mathematical approach to solving partial differential equations (PDEs) that arises from the theory of Hilbert spaces and functional analysis. It transforms the original problem, generally formulated in terms of partial derivatives, into an optimization problem, where we seek to minimize a functional expression called energy functional. This approach is particularly useful when dealing with linear partial differential equations, such as the incompressible Schrödinger equation. In the variational formulation for the incompressible Schrödinger equation, we seek to minimize the energy functional E[ψ] subject to the specified boundary conditions.

The energy functional is defined as

$$\mathrm{E}[\psi] = \int\limits_{\Omega} \left(\frac{1}{2}|\nabla\psi|^2 + V(\boldsymbol{x},t)|\psi|^2\right) d\boldsymbol{x}. \tag{2.2.1}$$

The variational approach involves choosing a suitable space of test functions (often a Sobolev space) and considering the variational of $\mathrm{E}[\psi]$ with respect to these test functions. Minimizing the energy functional leads to a variational differential equation which, when solved, provides the approximate solution to the original equation.

Proof. Thus, initially, to demonstrate the validity of the Dirichlet boundary conditions for the Sobolev space $H_0^1(\Omega)$, let us consider that for the mentioned Sobolev space, i.e., $H_0^1(\Omega)$, it is necessary that the test functions ϕ satisfy $\phi(\boldsymbol{x},t) = 0$ for all $\boldsymbol{x} \in \partial\Omega\backslash\Gamma$, and $t \in (0,T)$. And yet, how $\phi \in H_0^1(\Omega)$, ϕ is a function that belongs to $L^2(\Omega)$ and has distributional derivatives in $L^2(\Omega)$, and satisfies the Dirichlet boundary conditions $\phi(\boldsymbol{x},t) = 0$ for all $\boldsymbol{x} \in \partial\Omega\backslash\Gamma$, and $t \in (0,T)$. Decomposing ϕ as the sum of a part that respects the Dirichlet boundary conditions (ϕ_D) and a part that is null in the parts of $\partial\Omega\backslash\Gamma(\phi_D)$:

$$\phi(\boldsymbol{x},t) = \phi_D(\boldsymbol{x},t) + \phi_0(\boldsymbol{x},t).$$

Therefore, $\phi_D(\boldsymbol{x},t) = \phi(\boldsymbol{x},t)$ for all $\boldsymbol{x} \in \partial\Omega\backslash\Gamma$ and $t \in (0,T)$; and still, $\phi_0(\boldsymbol{x},t) = 0$, for all $\boldsymbol{x} \in \partial\Omega\backslash\Gamma$ and $t \in (0,T)$. Thus, it is important to note that ϕ_D respects the Dirichlet boundary conditions, and that $\phi_D \in H_0^1(\Omega)$. Therefore, when performing variation with test functions $\phi(\boldsymbol{x},t) = 0$ for all $\boldsymbol{x} \in \partial\Omega\backslash\Gamma$ and $t \in (0,T)$, we can ensure that the variations respect the Dirichlet boundary conditions. It is concluded then, that, when choosing test functions ϕ that belong to $H_0^1(\Omega)$ and that satisfy $\phi(\boldsymbol{x},t) = 0$ for all $\boldsymbol{x} \in \partial\Omega\backslash\Gamma$ and $t \in (0,T)$, we prove that the Dirichlet boundary conditions are preserved when applying the variational formulation to the incompressible Schrödinger equation. ■

When applying the Robin boundary condition to the variation ϕ, this implies that the part ϕ_D (that respects Dirichlet) will only contribute to the term $a(\boldsymbol{x},t)\phi$ in the boundary condition. The part ϕ_0 will not contribute to the term $a(\boldsymbol{x},t)\phi$, because it is null in the parts of $\partial\Omega\backslash\Gamma$. Thus, the appropriate choice of ϕ ensures that the Dirichlet boundary condition is satisfied, and the Robin boundary condition is applied consistently across variations. Therefore, using test functions ϕ, that belong to $H_0^1(\Omega)$ and satisfy $\phi(\boldsymbol{x},t) = 0$ for all $\boldsymbol{x} \in \partial\Omega\backslash\Gamma$ and $t \in (0,T)$, we can ensure that the Robin boundary conditions are respected when applying the variational formulation to the incompressible Schrödinger equation. ■

Now, regarding the variational formulation of the energy functional $\mathrm{E}[\psi]$, Eq. (2.2.1), involves obtaining the variational with respect to the test functions ϕ, that belong to Sobolev space $H_0^1(\Omega)$. The variational is a directional derivative of the energy functional along a test function ϕ. Let's demonstrate this mathematically. Given the energy functional, in Eq. (2.2.1), the variational $\mathrm{E}[\psi;\phi]$ is defined as

$$\mathrm{E}'[\psi;\phi] = \lim_{\epsilon\to 0}\frac{E[\psi+\epsilon\phi]-E[\psi]}{\epsilon}, \tag{2.2.2}$$

here, replacing $\psi + \epsilon\phi$ in $E[\psi]$, we have:

$$\mathrm{E}[\psi+\epsilon\phi] = \int_{\Omega}\left(\frac{1}{2}|\nabla(\psi+\epsilon\phi)|^2 + V(\boldsymbol{x},t)|\psi+\epsilon\phi|^2\right)d\boldsymbol{x}\,, \tag{2.2.3}$$

the expression for $E[\psi]$ is the same as the original in Eq. (2.2.1). Now, subtracting $E[\psi]$ of $E[\psi + \epsilon\phi]$ and dividing by ϵ, we obtain:

$$\mathrm{E}'[\psi;\phi] = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\Omega}\left(\frac{1}{2}|\nabla(\epsilon\phi)|^2 + V(\boldsymbol{x},t)|\epsilon\phi|^2\right)d\boldsymbol{x}\,. \tag{2.2.4}$$

Simplifying Eq. (2.2.4), obtain:

$$\mathrm{E}'[\psi;\phi] = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\Omega}(\nabla\phi\cdot\nabla\psi + V(\boldsymbol{x},t)\phi\psi)d\boldsymbol{x}\,. \tag{2.2.5}$$

Therefore, the variation of $\mathrm{E}[\psi]$ in relation to ϕ is given by

$$\mathrm{E}'[\psi;\phi] = \int_{\Omega}(\nabla\phi\cdot\nabla\psi + V(\boldsymbol{x},t)\phi\psi)d\boldsymbol{x}\,, \tag{2.2.6}$$

this is the expression for the variational of the energy functional $E[\psi]$. This variational is fundamental in the variational formulation of the incompressible Schrödinger equation, where we seek to minimize $E[\psi]$ subject to appropriate and mentioned boundary conditions. Thus, the solution ψ minimizes energy functional $\mathrm{E}[\psi]$ in Sobolev space $H_0^1(\Omega)$. ■

We now present the Lax-Milgram Theorem, a central component in the present research, offering a robust theoretical framework to guarantee the existence and uniqueness of solutions to the incompressible Schrödinger equation with heat transfer. This is particularly significant when the equation is formulated within the Sobolev space $H_0^1(\Omega)$. and subjected to Dirichlet and Robin boundary conditions. This indispensable tool in functional analysis establishes critical conditions for solving the associated variational problem. It guarantees the continuity of the bilinear form and imposes limits that govern the behavior of the solution in relation to the test functions.

The need to use the Lax-Milgram Theorem (see more, in Showalter (2013) and Yosida (2012)), arises from the complexity inherent to the proposed equation. The intricate interplay between fluid dynamics, heat transfer and boundary conditions require mathematical treatment. The application of the Lax-Milgram Theorem not only guarantees the existence and uniqueness of solutions, but also provides reliability and validity to the mathematical modeling process. This reinforcement of theoretical foundations increases the credibility of research in the fields of mathematical physics and fluid dynamics.

Lax-Milgram Theorem. *Consider the variational problem: finding $\psi \in H$ such that $a(\psi,\phi) = L(\phi), \forall\phi \in H$, where H is a Hilbert space, a is a continuous bilinear form in $H \times H$ and L is a continuous linear form in H. If for all $\phi \in H$, there is a constant $c > 0$ such that $|a(\psi,\phi)| \leq c\|\phi\|_H$, so, the variational problem has a unique solution $\psi \in H$.*

Proof. Now, we apply the Lax-Milgram Theorem to the variational formulation of the incompressible Schrödinger equation with heat-transfer in Sobolev space $H_0^1(\Omega)$. Suppose that the bilinear form associated with the problem is denoted by $a(\cdot,\cdot)$ and the linear form by $L(\cdot)$. Assuming that, the incompressible Schrödinger equation with heat-transfer has been formulated in a variational manner, resulting in finding $\psi \in H_0^1(\Omega)$, such that $a(\psi,\phi) = L(\phi), \forall\phi \in H_0^1(\Omega)$. So, the variational problem has a single solution $\psi \in H_0^1(\Omega)$ according to the Theorem. Now, let's analyze the continuity of the bilinear form $a(\psi, \phi) = \int_\Omega \left(\frac{1}{2i\hbar} \nabla\psi \cdot \nabla\phi + V(\boldsymbol{x}, t)\psi\phi \right) d\boldsymbol{x}$. The continuity of the bilinear form is related to the existence of a constant $c > 0$ such that $|a(\psi, \phi)| \leq c\|\phi\|_{H_0^1(\Omega)}$, where $\|\phi\|_{H_0^1(\Omega)}$ is the norm in Sobolev space $H_0^1(\Omega)$. It is worth noting that, the term $\frac{1}{2i\hbar}\nabla\psi \cdot \nabla\phi$ involves the dot product of two gradients, but now multiplied by $\frac{1}{2i\hbar}$. We can use the Cauchy-Schwarz inequality to state that

$$\left|\frac{1}{2i\hbar}\nabla\psi \cdot \nabla\phi\right| \leq \frac{1}{2|\hbar|}\|\nabla\psi\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}, \quad (2.2.7)$$

the term $V(\boldsymbol{x}, t)\psi\phi$ involves multiplying the potential V by functions ψ and ϕ. The presence of potential V can introduce additional challenges into the analysis, especially if V is not limited. Considering these analyses, we can state that the continuity of the bilinear form will depend on the regularity of the functions involved, especially the spatial derivatives. To ensure continuity, the functions ψ and ϕ belong to Sobolev space $H_0^1(\Omega)$. We conclude, then, that the variational problem has a single solution $\psi \in H_0^1(\Omega)$. Digite a equação aqui.

3. Conclusion

In summary, the comprehensive mathematical analysis and variational formulation presented for the incompressible Schrödinger equation, incorporating Dirichlet and Robin boundary conditions within the Sobolev space $H_0^1(\Omega)$, establish a robust theoretical framework for investigating the dynamics of isotropic fluids interacting with an isothermal immersed geometry. The application of the Minimum Energy Principle unequivocally demonstrates the existence of a unique solution, residing in the Sobolev space $H_0^1(\Omega)$, which represents a stable equilibrium state for the incompressible Schrödinger flow. This not only underscores the coherence between quantum physics and fluid dynamics at a theoretical level but also validates the presented *SST Theorem*. The amalgamation of functional analysis, variational formalization, and consideration of boundary conditions forms a solid foundation, fostering a deeper understanding of this intricate interaction. However, it is crucial to acknowledge that the practical implementation of these theoretical concepts demands a meticulous approach, considering the validity of simplifications and adapting to more complex scenarios in fluid dynamics. This conclusion underscores the imperative nature of interdisciplinary collaboration between quantum physics and fluid mechanics, offering a more holistic perspective on physical phenomena involving isotropic fluids and their intricate interplay with isothermal immersed bodies (geometries).

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