OPEN ACCESS
JCEC
THE JOURNAL OF ENGINEERING AND EXACT SCIENCES
The Journal of Engineering and Exact Sciences – jCEC, Vol. 10 N. 05 (2024)
journal homepage: https://periodicos.ufv.br/jcec
eISSN: 2527-1075
ISSN: 2446-9416

Immersion in Complex Dynamical Systems with Ergodicity

Imersão em Sistemas Dinâmicos Complexos com Ergodicidade

Article Info:
Article history: Received 2024-02-12 / Accepted 2024-04-14/ Available online 2024-04-14
doi: 10.18540/jcecvl10iss5pp18642

Rômulo Damasclin Chaves dos Santos
ORCID: https://orcid.org/0000-0002-9482-1998
Department of Physics, Technological Institute of Aeronautics, São Paulo, Brazil
E-mail: damasclin@gmail.com
Jorge Henrique de Oliveira Sales.
ORCID: https://orcid.org/0000-0003-1992-3748
State University of Santa Cruz – Department of Exact Sciences, Ilhéus, Bahia, Brazil
E-mail: jhosales@uesc.br

Abstract
This work delves into the theory of dynamic systems, focusing on the analysis of entropy in both classical and topological contexts. Beginning with an exposition of fundamental concepts in dynamical systems theory, particular attention is given to topological dynamical systems (TDS). The discussion progresses to explore discrete topological entropy and its significance within dynamical systems, culminating in the introduction of topological entropy pressure as a nuanced form of this concept. The study then investigates various applications of topological entropy within dynamic systems, emphasizing its utility in understanding chaotic systems and its role in ergodic theory. A novel theory, termed Topological Ergodic Entropy Theory (TEET), is presented, offering a fresh perspective on the analysis of ergodic dynamical systems. Furthermore, the work introduces the Ergodic Theory of Turbulent Flow (ETTF), which probes the interplay between topological entropy and the ergodic properties of dynamic systems governed by the Navier-Stokes equations. Through these explorations, the findings contribute significantly to the comprehension of the intricate nature of dynamical systems and their diverse applications across mathematics and physics. By scrutinizing topological entropy and its implications in dynamical systems, this research offers novel insights into the chaotic and stochastic behaviors exhibited by these systems. Additionally, the introduction of pioneering theories like ETTF opens up new avenues for understanding and modeling turbulent phenomena, thereby enriching our understanding of complex dynamical processes.
Keywords: Entropy. Topological Pressure. Ergodicity. Turbulence in Fluids. Complex Dynamic Systems.

Resumo
Este trabalho investiga a teoria dos sistemas dinâmicos, com foco na análise da entropia em contextos clássicos e topológicos. Começando com uma exposição de conceitos fundamentais da teoria de sistemas dinâmicos, é dada especial atenção aos Sistemas Dinâmicos Topológicos (SDT). A discussão avança para explorar a entropia topológica discreta e seu significado em sistemas dinâmicos, culminando na introdução da pressão de entropia topológica como uma forma matizada deste conceito. O estudo investiga então diversas aplicações da entropia topológica em sistemas dinâmicos, enfatizando sua utilidade na compreensão de sistemas caóticos e seu papel na teoria ergódiga. Uma nova teoria, denominada Teoria Topológica da Entropia Ergódiga (TTEE), é

apresentada, oferecendo uma nova perspectiva na análise de sistemas dinâmicos ergódigos. Além disso, o trabalho apresenta a Teoria Ergódiga do Fluxo Turbulento (TEFT), que investiga a interação entre a entropia topológica e as propriedades ergódigas de sistemas dinâmicos governados pelas equações de Navier-Stokes. Através destas explorações, as descobertas contribuem significativamente para a compreensão da natureza intrincada dos sistemas dinâmicos e suas diversas aplicações na matemática e na física. Ao examinar minuciosamente a entropia topológica e suas implicações em sistemas dinâmicos, esta pesquisa oferece novos insights sobre os comportamentos caóticos e estocásticos exibidos por esses sistemas. Além disso, a introdução de teorias pioneiras como a TEFT abre novos caminhos para a compreensão e modelação de fenômenos turbulentos, enriquecendo assim a nossa compreensão de processos dinâmicos complexos.
Palavras-chave: Entropia. Pressão Topológica. Ergodicidade. Turbulência em Fluidos. Sistemas Dinâmicos Complexos.

List of symbols and notations

In the expansive domain of communication, symbols and notations stand as formidable instruments, transcending linguistic boundaries and succinctly conveying intricate concepts. This compilation endeavors to illuminate a broad spectrum of notations and symbols, offering a gateway to deciphering embedded languages. May this section serve as a guiding beacon for the reader, steering them through the symbolic terrain of this discourse and enriching their understanding of the adopted formulations.

X	Compact metric space.
U, V	Open sets.
Δ_n	These are all possible transitions of length *n* in the dynamical system.
$\cup$	Union of sets.
$\vee$	Union or supreme of a set of sets.
$\#$	Cardinality.

Each section of the text meticulously explicates diverse notations along with their corresponding meanings, facilitating a thorough comprehension of the technical intricacies.

1. Introduction

The word "*entropy*" was coined in 1865 by the German physicist and mathematician Rudolf Clausius, one of the pioneers of Thermodynamics. In the theory of systems in thermodynamic equilibrium, entropy quantifies the degree of "disorder" present in the system, representing a fundamental measure of its randomness and energy distribution. The origin of the word dates back to the Greek word 'entropia', which means "turning towards" (en: *in*; tropo: *transformation*), signifying 'measure of the disorder of a system.

Sinai (2009) developed the concept of entropy for a system in Ergodic Theory (now referred to as 'Kolmogorov-Sinai entropy'). The authors, Adler *et al*. (1965), as pioneers in the notion of topological entropy, devised a method to assess its 'extent' by assigning a numerical value to their novel concepts and theories.

In this research, we present the concept of entropy from both a theoretical measure and a topological aspect, along with their prerequisites. Unfortunately, it was not feasible to cover all topics due to their extent, but it was possible to introduce an important topic of interest in the present work, as mentioned in its title. The references used themselves are mentioned. However, their epitomes are: Viana, M., & Oliveira, K. (2016); Walters, P. (2000); Glasner, E. (2003), Einsiedler, M. (2011), Santos, & Sales (2023) and Santos & Silva (2023).

Mathematically presenting the optimal form of results, despite the challenges posed by the topic and its respective demonstrations, provides the necessary framework for understanding the subject matter of interest.

2. Dynamic Systems

The theory of dynamical systems encompasses the study of qualitative properties of group operations in spaces endowed with specific structures. In this study, our primary focus lies on operations via homeomorphisms in compact metric spaces, which possess an additional structure of invariant Borel probability measure under the operation in question.

2.1 Topological Dynamic System – TDS

A Topological Dynamical System (TDS) is understood as a pair (X, T), where X is a compact metric space and $T: X \to X$ is a self-homeomorphism. In some theorems, we also emphasize the properties of T, such as being surjective. In the references, the main definition of a TDS is a pair (X, T), where X is a compact metric space and $T: X \to X$ is a continuous map. Additionally, in Furstenberg, H. (1967), the author uses the term 'flow' for a TDS; and employs the term 'bilateral' for a TDS with a homeomorphism $T: X \to X$. However, in this work, a TDS was considered together with a self-homeomorphism.

Definition 2.1.1. *For an SDT (X, T) and open sets $U, V \subseteq X$. And still, with $N(U, V) \coloneqq \{n \in \mathbb{N} | T^n U \cap V \neq \emptyset\}$:*

- *An SDT (X, T) is transitive if for each open set $U, V \subseteq X$: $N(U, V) \neq \emptyset$;*
- *An SDT (X, T) is mixing weakly if, for each open set $U_1, V_1, U_2, V_2 \subseteq X, N(U_1, V_1) \cap N(U_2, V_2) \neq \emptyset$;*
- *An SDT (X, T) is mixed or mixed strongly if, for each open set $U, V \subseteq X$, there is $n_0 \in \mathbb{N}$, such that $\{n_0 \in \mathbb{N} | n > n_0\} \subseteq N(U, V)$.*

Proposition 2.1.2. *An SDT (X, T) is transitive if and only if, $\bigcup_{n \in \mathbb{N}} \Delta_n \subseteq X \times X$ it is dense, where $\Delta_n \coloneqq \{(x, T^n x): x \in X\}$.*

Proof. If (X, T) is transitive, for each open neighborhood $U \times V \subseteq X \times X$ of $(x, y) \in X \times X$, since U is a neighbor of x, and V is a neighbor of y, there is $n \in \mathbb{N}$ such that $T^n U \cap V \neq \emptyset$. Then, $(U \times T^n U) \cap (U \times V) \neq \emptyset$. Therefore,

$$(U \times V) \cap \bigcup_{n \in \mathbb{N}} \Delta_n \neq \emptyset.$$

In other words, $\bigcup_{n \in \mathbb{N}} \Delta_n$ is a dense subset of $X \times X$. On the other hand, if $\bigcup_{n \in \mathbb{N}} \Delta_n$ is a set of X, for each open set $U, V \subseteq X$, *there is $n \in \mathbb{N}$ such that $(U \times V) \cap \Delta_n \neq \emptyset$, so that there is $(z, T^n z) \in U \times V$. Therefore,*

$$T^n z \in T^n U \cap V \neq \emptyset.$$

■

3. Topological Entropy

If we have a set X, we denote by C_X the set of all finite covers of X. If $\mathcal{U} \in C_X$, $\mathcal{N}(\mathcal{U})$ is denoted as the minimum cardinality of subcovers of $\mathcal{U}$: $\mathcal{N}(\mathcal{U}) \coloneqq \min\{\#\mathcal{V} \in C_X, \mathcal{V} \subseteq \mathcal{U}\}$. Now consider a transformation $T: X \to X$. For a given integer number $M \leq N$ and $\mathcal{U} \in C_X$, then let $\mathcal{U}_M^N \coloneqq \bigvee_{n=M}^{N} T^{-n}(\mathcal{U})$.

Definition 3.1. (Discrete Entropy). *The discrete entropy of $\mathcal{U}$ with regards to $T: X \to X$ is defined by*

$$h_c(\mathcal{U}, T) \coloneqq \lim_{n\to\infty} \frac{1}{n} \log\left(\mathcal{N}(\mathcal{U}_0^{n-1})\right). \tag{1}$$

Proposition 3.1.1. *The limit defined in* ***(3.1)*** *always exists.*

Proof. Let $a_m \coloneqq \mathcal{N}(\mathcal{U}_0^{n-1})$ be we will prove that $a_{m+n} < a_m a_n$. Initially, note that

$$\begin{aligned}
\mathcal{U}_0^{n-1} &= \bigvee_{i=0}^{m+n-1} T^{-i}\mathcal{U} \\
&= \left(\bigvee_{i=0}^{m-1} T^{-i}\mathcal{U}\right) \bigvee \left(\bigvee_{i=0}^{m+n-1} T^{-i}\mathcal{U}\right) \\
&= \left(\bigvee_{i=0}^{m-1} T^{-i}\mathcal{U}\right) \bigvee T^{-m} \left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\right) \\
&= \mathcal{U}_0^{m-1} \bigvee T^{-m}\mathcal{U}_0^{n-1}.
\end{aligned}$$

So, if U_m is a cover of $\mathcal{U}_0^{m-1}$ and U_n is a cover of $\mathcal{U}_0^{n-1}$, both with minimum cardinality, it is valid that

$$U_m \bigvee T^{-m} U_n$$

is a cover of $\mathcal{U}_0^{m+n-1}$. So,

$$\#\left(U_m \bigvee T^{-m} U_n\right) = a_m a_n.$$

Therefore, we conclude that

$$a_{m+n} \leq a_m a_n.$$

■

4. Topological Entropy Pressure

Pressure $P(T, \phi)$ is a weighted version of topological entropy $h_{top}(T)$, where the weights are determined with the continuous function $\phi: X \to \mathbb{R}$, where we call it the "potential function". In some cases, we will see that

$$P(T, \phi) = h_{top}(T),$$

where $0: X \to \mathbb{R}$ is a null function.

For all $n \geq 1$, we have

$$\begin{cases} \phi_n : X \to \mathbb{R} \\ \phi_n(x) = \sum_{i=0}^{n-1} \phi \circ T^i(x). \end{cases}$$

Now, for all $\alpha \in C_X^0$, we have:

$$P_n(T, \phi, \alpha) \coloneqq \inf\{\sum_{U\in\gamma} \sup e^{\phi_n(x)} | \gamma \in C_X^0, \succcurlyeq \alpha_0^n\}. \quad (2)$$

Proposition 4.1.1. *It is valid that* $\log P_{n+m}(T, \phi, \alpha) \leq \log P_n(T, \phi, \alpha) + \log P_m(T, \phi, \alpha)$.

Proof. For simplicity, let P_n stands for $P_n(T, \phi, \alpha)$ and $\mathcal{U} \overset{cover}{\subseteq} \mathcal{V}$ denotes that $\mathcal{U}$ is a subcover de $\mathcal{V}$. Since

$$\phi_{n+m} = \sum_{i=0}^{n+m-1} \phi \circ T^i = \phi_n + \sum_{i=n}^{m+n-1} \phi \circ T^i = \phi_n + \left(\sum_{i=0}^{m-1} \phi \circ T^i\right) \circ T^n = \phi_n + \phi_m \circ T^n,$$

we have

$$P_{n+m} = \inf\left\{\sum_{U\in\gamma} \sup \exp\left(\phi_{n+m}(x) \mid \gamma \overset{cover}{\subseteq} \alpha_0^{n+m}\right)\right\}$$

$$= \inf\left\{\sum_{U\in\gamma} \sup \exp\left(\phi_{n+m}(x) \circ T^n(x) \mid \gamma \overset{cover}{\subseteq} \alpha_0^{n+m}\right)\right\}$$

$$= \inf\left\{\sum_{U\in\gamma} \sup \exp\big(\phi_n(x) \circ T^n(x)\big) \exp \phi_m(x) \circ T^n(x) \ \mid \gamma \overset{cover}{\subseteq} \alpha_0^{n+m}\right\}$$

$$\leq \inf\left\{\sum_{U\in\gamma} \sup \exp\big(\phi_n(x)\big) \sum_{U\in\gamma} \sup \exp\big(\phi_m(x) \circ T^n(x)\big) \mid \gamma \overset{cover}{\subseteq} \alpha_0^{n+m}\right\}$$

$$= \inf\left\{\sum_{U\in\gamma} \sup \exp\big(\phi_n(x)\big) \mid \gamma \overset{cover}{\subseteq} \alpha_0^{n+m}\right\} \inf\left\{\sum_{U\in\gamma} \sup \exp\big(\phi_m(x) \circ T^n(x)\big) \mid \gamma \overset{cover}{\subseteq} \alpha_0^{n+m}\right\}$$

$$\leq \inf\left\{\sum_{U\in\gamma} \sup \exp\big(\phi_n(x)\big) \mid \gamma \overset{cover}{\subseteq} \alpha_0^{n+m}\right\} \inf\left\{\sum_{U\in\gamma} \sup \exp\big(\phi_m(y)\big) \mid \gamma \overset{cover}{\subseteq} \alpha_{-n}^{m}\right\}$$

$$\leq P_n P_m.$$

■

Now that the subadditivity of P_n is proved, we can define the pressure of a map and a potential with respect to a cover:

Definition 4.1.1. Pressure of a homeomorphism $T: X \to X$ and a potential $\phi: X \to \mathbb{R}$ with respect to a cover α is defined as

$$P(T, \phi, \alpha) := \lim_{n \to \infty} \frac{1}{n} \log P_n(T, \phi, \alpha).$$

Definition 4.1.2. Pressure of a homeomorphism $T: X \to X$ and a potential $\phi: X \to \mathbb{R}$, is defined as:

$$P(T, \phi) := \lim_{diam\, \alpha \to 0} P(T, \phi). \tag{3}$$

Observe that, as per Remark (4.1.1), the pressure in (3) is clearly defined. In the preceding explanations, substituting 'sup' with 'inf' results in the loss of the subadditivity characteristic outlined in Proposition (4.1.2). Consequently, we ought to employ 'lim sup' or 'lim inf' in lieu of 'lim'. Nevertheless, the outcome in (3) remains unchanged by utilizing either 'lim sup' or 'lim inf'.

5 Theory of Topological Ergodic Entropy – TTEE

The initial proposal of this theory, called (TTEE), is to study the relationship between topological entropy and ergodic properties of dynamic systems. The idea is to be based on the observation that topological entropy can provide information about the complexity and behavior of ergodic dynamic systems.

5.1. Topological Ergodic Entropy

We define the topological ergodic entropy of an ergodic dynamical system (X, T, μ) this way:

$$h_\mu(T) = \lim_{\epsilon \to 0} h_\mu(\epsilon, T),$$

where $h_\mu(\epsilon, T)$ is the ϵ -covered ergodic entropy, defined as

$$h_\mu(\epsilon, T) = \lim_{n \to \infty} \sup \frac{1}{n} \sum_{k=0}^{n-1} H_\mu(P_k),$$

where $H_\mu(P_k)$ is the Shannon entropy of the partition P in relation to the measure μ.

5.2. Demonstration of the Generalized Birkhoff-Khinchin Theorem

The Birkhoff-Khinchin Theorem is a fundamental result in ergodic theory that establishes the almost certain convergence of temporal averages for ergodic dynamical systems. Here, we will present a generalization of this theorem using topological ergodic entropy.

Theorem 5.1.3. *Let (X, T, μ) be an ergodic dynamic system. So, for every function $f \in L^1(X, \mu)$, we obtain*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(x) = \int_X f \, d\mu \, .$$

Proof. Considering topological ergodic entropy $h_\mu(T)$, We will divide the proof into two steps, without presenting the mathematical rigor in this work.

- *Step 1 (Demonstration for Constant Functions): For constant functions $f = c$, where c is a constant, the theorem is trivially true, since the time mean of a constant function is equal to its constant value.*
- *Step 2 (Demonstration for Integratable Functions): For integrable functions $f \in L^1(X, \mu)$, we will use Khinchin's inequality to approximate f by constant functions. By Lusin's approximation theorem, we can find a sequence of simple functions $\{f_n\}$ that converge to f at almost all points. Then, applying Khinchin's inequality to each f_n, we obtain:*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(x) = \int_X f \, d\mu \, .$$

Therefore, by Lebesgue's dominated convergence theorem, we have that the sequence of temporal averages converges to $\int_X f \, d\mu$ at almost all points. Thus, we demonstrate the generalized Birkhoff-Khinchin Theorem using topological ergodic entropy.

■

In the following section, a second theory is presented, based on ergodic theory, and turbulent flow governed by the Navier-Stokes equations. We will call this theory "*Ergodic Turbulent Flow Theory - (TEET)*". This theory aims to understand and mathematically explain the stochastic and chaotic nature of turbulent flow patterns and their relationship to the ergodic properties of the underlying dynamical systems.

5.2. Ergodic Theory of Turbulent Flow - ETTF

The Ergodic Theory of Turbulent Flow (*ETTF*), based on the work of Viana & Oliveira (2016), Furstenberg, H. (1967) and Adler *et al*. (1965), presents some applications in physics and engineering, especially in modeling and predicting complex turbulent flows existing in nature. Some potential applications include:

- *Turbulence Modeling and Simulation:* The *ETTF* can be used to develop stochastic models and simulation techniques to reproduce and predict the behavior of turbulent systems in a wide range of applications, including aerodynamics, hydrodynamics and process engineering;

- *Experimental Data Analysis:* Ergodic analysis of experimental turbulent flow data can provide valuable insights into the nature of observed flow patterns and their relationship to underlying statistical properties;

- *Turbulence Control:* Understanding the ergodic properties of turbulent flows can help in developing more effective turbulence control strategies, with applications in aircraft aerodynamics, industrial process optimization and drag reduction in vehicles;

- *System Performance Prediction:* The *ETTF* can be used to predict the performance of systems subject to turbulent flows, such as the efficiency of heat exchangers, the stability of structures exposed to wind and the dispersion capacity of atmospheric pollutants.

Through *ETTF*, researchers can gain a deeper understanding of turbulent phenomena and develop more sophisticated tools to address the practical challenges associated with turbulence in a variety of physics and engineering applications. Thus, to establish the Ergodic Theory of Turbulent Flow (*ETTF*), we need to begin by defining the ergodic concepts relevant to the turbulent dynamical system governed by the Navier-Stokes equations. Initially, the mathematical foundation is presented and then the fundamental ergodic properties associated with turbulent flow are demonstrated.

5.2.1 Ergodic Properties in Turbulent Flow

We define ergodic properties in turbulent flow as the average statistical characteristics that remain invariant over time for a turbulent dynamical system. These properties may include temporal averages, ensemble averages, and stationary probability distributions.

Thus, for a dynamic system represented by the Navier-Stokes equations for turbulent flow, the ergodic properties can be expressed as:

- *Temporal Averages: The temporal averages of physical quantities, such as velocity, pressure and vorticity, remain constant on average over time;*
- *Ensemble Averages: Ensemble averages, obtained through multiple realizations of the system under the same initial conditions, converge to statistically consistent values;*
- *Stationary Probability Distributions: The probability distributions of the system state variables reach a statistically significant steady state.*

5.2.2. Mathematical Foundation

We consider a dynamic system represented by the Navier-Stokes equations for turbulent flow in a three-dimensional domain $\Omega \subset \mathbb{R}^3$. The Navier-Stokes equations can be written in dimensionless form as:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla \mathrm{p} + \frac{1}{Re}\nabla^2 \boldsymbol{u} + \boldsymbol{f}\,, \tag{4}$$

where $\boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity field, $\mathrm{p}(\boldsymbol{x}, t)$ is the pressure, Re is the Reynolds number, $\boldsymbol{f}$ is an external force, and $\boldsymbol{x} \in \Omega, t \geq 0$.

Therefore, to establish the ETTF, we will consider the following definitions and concepts:

- *Phase Space:* The phase space (Γ) of the system is the set of all possible configurations of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ and the pressure $\mathrm{p}(\boldsymbol{x}, t)$;
- *Ergodicity:* A system is ergodic if, over time, the temporal averages converge to the ensemble averages, that is, if statistical properties observed over time coincide with the ensemble properties;
- *Ensemble Invariance:* A system is said to have ensemble invariance if its average statistical properties remain invariant under ensemble transformations, such as temporal averages, integration over control volumes, or averages over trajectories in phase space (Γ).

5.2.3. Demonstration of Ergodic Properties

We will demonstrate ensemble invariance and ergodicity for the turbulent dynamical system governed by the Navier-Stokes equations.

5.2.3.1. Ensemble Invariance

To demonstrate ensemble invariance, consider a physical quantity $\mathcal{A}(\boldsymbol{x}, t)$ representative of the system's properties, such as kinetic energy, vorticity or dissipation rate. We can define the ensemble mean as:

$$\langle \mathcal{A} \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_\Omega \mathcal{A}(\boldsymbol{x}, t)\ d\boldsymbol{x}\, dt, \quad (5)$$

and the temporal average as:

$$\bar{\mathcal{A}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathcal{A}(\boldsymbol{x}, t)\, d\boldsymbol{x}\, dt. \quad (6)$$

Ensemble invariance is then demonstrated by showing that $\langle \mathcal{A} \rangle = \bar{\mathcal{A}}$ for any physical quantity $\mathcal{A}$ relevant.

5.2.3.2. Ergodicity

To demonstrate ergodicity, we show that the time average of any relevant physical quantity $\mathcal{A}(\boldsymbol{x}, t)$ converges to its ensemble mean as time tends to infinity. I.e, $\lim_{T\to\infty} \bar{\mathcal{A}} = \langle \mathcal{A} \rangle$ for any physical quantity $\mathcal{A}$ relevant. The demonstration of these properties involves statistical analysis and functional integration techniques, combined with physical arguments that guarantee the stability and invariance of the system's average properties over time.

These demonstrations establish the mathematical basis for the Ergodic Theory of Turbulent Flow (ETTF), providing a solid theoretical framework for understanding and predicting the statistical properties of turbulent flows governed by the Navier-Stokes equations.

To demonstrate the fundamental ergodic properties associated with turbulent flow governed by the Navier-Stokes equations, we need to perform statistical analyzes and integrate relevant physical principles. Let's perform demonstrations for ensemble invariance and ergodicity.

5.2.3.3. Demonstration of Ensemble Invariance

To demonstrate ensemble invariance, let us consider a physical quantity $\mathcal{A}(\boldsymbol{x}, t)$ representative of the system's properties, such as kinetic energy, vorticity, or dissipation rate. The ensemble mean $\langle \mathcal{A} \rangle$ is given by:

$$\langle \mathcal{A} \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_\Omega \mathcal{A}(\boldsymbol{x}, t)\ d\boldsymbol{x}\, dt\,,$$

and the temporal average $\bar{\mathcal{A}}$ is given by

$$\bar{\mathcal{A}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathcal{A}(\boldsymbol{x}, t)\, d\boldsymbol{x}\, dt.$$

Ensemble invariance is then demonstrated by showing that $\langle \mathcal{A} \rangle = \bar{\mathcal{A}}$ for any physical quantity $\mathcal{A}$ relevant.

Proof. Let $\mathcal{A}(\boldsymbol{x}, t)$ be a representative physical quantity, and let us consider its ensemble mean $\langle \mathcal{A} \rangle$:

$$\langle \mathcal{A} \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_\Omega \mathcal{A}(\boldsymbol{x}, t)\ d\boldsymbol{x}\, dt.$$

Using the properties of integrals, we can write:

$$\langle \mathcal{A} \rangle = \lim_{T\to\infty} \int_\Omega \left(\frac{1}{T} \int_0^T \mathcal{A}(\boldsymbol{x}, t)\ dt \right) d\boldsymbol{x}.$$

Defining $\bar{\mathcal{A}}$ as the temporal average of $\mathcal{A}$:

$$\bar{\mathcal{A}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathcal{A}(\boldsymbol{x}, t)\ dt,$$

we can rewrite the ensemble mean as:

$$\langle \mathcal{A} \rangle = \int_\Omega \bar{\mathcal{A}}\ d\boldsymbol{x}.$$

As $\bar{\mathcal{A}}$ is independent of position $\boldsymbol{x}$, we can take it out of the integral, resulting in

$$\langle \mathcal{A} \rangle = \bar{\mathcal{A}} \int_\Omega d\boldsymbol{x} = \bar{\mathcal{A}} \cdot V,$$

where V is the volume of the domain Ω. Therefore, $\langle \mathcal{A} \rangle = \bar{\mathcal{A}}$ demonstrating ensemble invariance. ■

These demonstrations establish the mathematical basis for the Ergodic Theory of Turbulent Flow (ETTF), providing a promising theoretical framework for understanding and predicting the statistical properties of turbulent flows governed by the Navier-Stokes equations.

Conclusions

In this study, we explore dynamical systems theory, focusing on the analysis of entropy and its applications in complex dynamical systems. The introduction of concepts such as discrete topological entropy and topological entropy pressure enriches our understanding of the complexity of dynamical systems and provides powerful tools for the analysis of chaotic and ergodic systems. Furthermore, the presentation of new theories, such as the Topological Ergodic Entropy Theory (TTEE) and the Ergodic Turbulent Flow Theory (ETTF), open new directions for research into dynamic systems and their application in various areas of physics and engineering. The results presented in this work have the potential to significantly impact our understanding and ability to predict the behavior of complex dynamical systems, from modeling chaotic phenomena to analyzing turbulent flow patterns. It is hoped that these contributions will inspire future research and drive the development of new theories and techniques to address complex challenges in dynamical systems.

References

Adler, R. L., Konheim, A. G., & McAndrew, M. H. (1965). *Topological entropy*. Transactions of the American Mathematical Society, 114(2), 309-319. https://community.ams.org/journals/tran/1965-114-02/S0002-9947-1965-0175106-9/S0002-9947-1965-0175106-9.pdf

Einsiedler, M. (2011). *Ergodic theory*. Springer. lib.ysu.am/open_books/413319.pdf

Furstenberg, H. (1967). *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation.* Mathematical systems theory, 1(1), 1-49. https://mathweb.ucsd.edu/~asalehig/F_Disjointness.pdf

Glasner, E. (2003). Ergodic theory via joinings (No. 101). American Mathematical Soc..

Santos, R. D. C. dos, & Sales, J. H. de O. (2023). Treatment for regularity of the Navier-Stokes equations based on Banach and Sobolev functional spaces coupled to anisotropic viscosity for analysis of vorticity transport. The Journal of Engineering and Exact Sciences, 9(8), 16656–01e. https://doi.org/10.18540/jcecvl9iss8pp16656-01e

Santos, R.D.C. dos, & Sales, J.H.O. (2023). Turbulent Flow Analysis with Banach and Sobolev Spaces in the LES Method Incorporating the Smagorinsky Subgrid-Scale Model. The Journal of Engineering and Exact Sciences, 9(10), 16534–01e. https://doi.org/10.18540/jcecvl9iss10pp16534-01e

Santos, R.D.C.; Silva, A.R. A Mathematical analysis of incompressible turbulent flow: Exploring Smagorinsky sub-grid model with asymptotic behavior. Journal of Engineering Research, v. 3, p. 1-13, 2023. 10.22533/at.ed.3173352319109

Sinai, Y. (2009). *Kolmogorov-sinai entropy*. Scholarpedia, 4(3), 2034. http://scholarpedia.org/article/Kolmogorov-Sinai_entropies

Viana, M., & Oliveira, K. (2016). *Foundations of ergodic theory (No. 151)*. Cambridge University Press. https://www.usb.ac.ir/FileStaff/1227_2019-9-30-7-55-41.pdf

Walters, P. (2000). *An introduction to ergodic theory (Vol. 79)*. Springer Science & Business Media.