OPEN ACCESS
The Journal of Engineering and Exact Sciences – jCEC, Vol. 10 N. 06 (2024)
journal homepage: https://periodicos.ufv.br/jcec
eISSN: 2527-1075
ISSN: 2446-9416

Turbulent System Dynamics: An Introduction to Spectral Decomposition and External Forcing in Sobolev Spaces

Dinâmica de Sistemas Turbulentos: Uma Introdução à Decomposição Espectral e Forçamento Externo em Espaços Sobolev

Article Info:
Article history: Received 2023-12-12 / Accepted 2024-05-15 / Available online 2024-05-15
doi: 10.18540/jcecvl10iss6pp18659

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Abstract
Turbulent systems represent a complex and ubiquitous phenomenon in various natural and engineered environments. This study investigates the dynamics of turbulent systems through the lens of four fundamental theorems. The first theorem establishes a spectral decomposition of the velocity correlation function, shedding light on the coherent and incoherent structures within turbulent flows. The second theorem delves into the energy distribution in Sobolev spaces, providing insights into the regularity and properties of turbulent solutions. Expanding on these findings, the third theorem explores the influence of external forcing on energy distribution, elucidating how external factors shape turbulent behavior. Building upon these insights, the fourth theorem integrates the concepts from the previous theorems, describing the interplay between spectral decomposition, energy distribution, and external forcing effects in turbulent systems. This comprehensive analysis offers a deeper understanding of turbulent dynamics, with implications for fields such as fluid mechanics, atmospheric science, and engineering. By elucidating the underlying mechanisms governing turbulent behavior, this study paves the way for improved modeling, prediction, and control of turbulent systems in various practical applications.
Keywords: Turbulent systems, Mixed convection, Spectral decomposition, Sobolev spaces.

Resumo
Os sistemas turbulentos representam um fenômeno complexo e onipresente em vários ambientes naturais e de engenharia. Este estudo investiga a dinâmica de sistemas turbulentos através das lentes de quatro teoremas fundamentais. O primeiro teorema estabelece uma decomposição espectral da função de correlação de velocidade, esclarecendo as estruturas coerentes e incoerentes dentro de fluxos turbulentos. O segundo teorema investiga a distribuição de energia em espaços de Sobolev, fornecendo insights sobre a regularidade e as propriedades de soluções turbulentas. Expandindo essas descobertas, o terceiro teorema explora a influência do forçamento externo na distribuição de energia, elucidando como os fatores externos moldam o comportamento turbulento. Com base nesses insights, o quarto teorema integra os conceitos dos teoremas anteriores, descrevendo a interação entre decomposição espectral, distribuição de energia e efeitos de forçamento externos em

sistemas turbulentos. Esta análise abrangente oferece uma compreensão mais profunda da dinâmica turbulenta, com implicações para campos como mecânica dos fluidos, ciência atmosférica e engenharia. Ao elucidar os mecanismos subjacentes que governam o comportamento turbulento, este estudo abre caminho para uma melhor modelagem, previsão e controle de sistemas turbulentos em diversas aplicações práticas.
Palavras-chave: Sistemas turbulentos, Convecção mista, Decomposição espectral, Espaços de Sobolev.

1. Introduction

Turbulent systems, such as turbulence in fluids, considering mixed convection (heat-transfer by convection), are ubiquitous in both natural phenomena and engineering applications, such as atmospheric flows, industrial processes, and environmental fluid dynamics. The intricate interplay between turbulence and heat-transfer in these flows significantly influences system performance optimization and the design of efficient heat transfer devices. Prior research has investigated various aspects of turbulence and heat-transfer interaction.

The complex interactions between turbulence and heat-transfer in these flows pose significant challenges and opportunities for research and engineering practice. Previous studies have investigated various aspects of turbulent heat transfer and convective flows. The works of Lumley (1967) and Patankar & Spalding (1974), explored turbulent mixing processes and their implications for heat-transfer in environmental fluid dynamics, highlighting the importance of understanding turbulent structures for accurate heat-transfer predictions. The authors Patankar & Spalding (1974) and Cant & Pope (2001), developed the highlighting the role of turbulent eddies in enhancing convective heat transfer rates and the k-epsilon turbulence model, which has been instrumental in simulating turbulent flows with heat transfer in engineering applications. These seminal works laid the foundation for further research into turbulence-heat transfer interactions, conducted studies on turbulent heat-transfer in pipe flows, revealing the complex relationship between turbulent intensity and convective heat-transfer coefficient.

Other more recent works, such as Santos & Sales (2023), the mathematical analysis applied in this work serves as a pillar for a broader investigation into the regularity of the Navier-Stokes Equations. In this context, this investigation marks a significant step in the advancement of the Smagorinsky model coupled to the LES methodology, resulting, based on Banach and Sobolev Spaces, in a new theorem that points the way to the construction of an anisotropic viscosity model (not yet discussed). In principle, the effort dedicated here aims to present a more comprehensive mathematical analysis, promoting a more level understanding of the challenge posed by the regularity of the Navier-Stokes equations.

Despite significant progress, there remain gaps in our understanding of turbulence and heat-transfer in mixed convection flows. Traditional approaches often fail to capture the intricate dynamics and spatial-temporal correlations present in turbulent flows with mixed convection. In this context, there is a need for innovative methodologies that can provide deeper insights into the underlying physics of these complex flows.

In response to this challenge, this study proposes a novel spectral decomposition approach to analyze the correlation functions of velocity and temperature fields in turbulent flows with mixed convection heat transfer. By leveraging techniques from functional analysis and Sobolev spaces, we aim to decompose these correlation functions into coherent and incoherent components, thereby unraveling the complex interplay between turbulence and heat transfer. Through this approach, we seek to advance our understanding of turbulent flows with mixed convection and pave the way for the development of more accurate turbulence models and heat-transfer prediction techniques.

2. Foundations of Turbulent Dynamics

2.1 Spectral Decomposition, Energy Distribution, and External Forcing

Turbulent systems represent one of the most challenging and intriguing phenomena in fluid dynamics, with broad implications across multiple scientific and engineering disciplines. Over the years, significant progress has been made in understanding turbulent systems, driven by theoretical developments and advances in computational techniques.

The key to unraveling the complexities of turbulence are fundamental concepts such as spectral decomposition, energy distribution and the influence of external forces. The work of Lumley (1967) provided initial insights into the spectral decomposition of turbulent velocity correlation functions, laying the foundation for subsequent research in this area. The authors Frisch (1995) and Cant & Pope (2001), further expanded these concepts, elucidating the role of coherent and incoherent structures in turbulent flows. At the same time Constantin & Foiaş (1988) and Caffarelli *et al.* (1982), contributed to our understanding of energy distribution in Sobolev spaces, highlighting the importance of regularity and properties of turbulent solutions.

Building on this foundation, the present study introduces new theorems that integrate these concepts, providing deeper insights into turbulent dynamics. By combining spectral decomposition, energy distribution in Sobolev spaces, and the influence of external forcings, our theorems offer a comprehensive framework for understanding and modeling turbulent systems. The following sections will detail each theorem and its implications, based on the rich literature in the field

The following section, presents an important theorem that will be discussed and demonstrated, seeking to present the mathematical foundations.

3. Spectral Decomposition of Velocity and Correlation Function in Dissipative Turbulence

3.1. ***Theorem 1.*** *For a fluid system experiencing dissipative turbulence, the velocity correlation function $\mathcal{R}(\boldsymbol{r},t)$ can be spectrally decomposed into the sum of a coherent component and an incoherent component,*

$$\mathcal{R}(\boldsymbol{r},t) = R_c(\boldsymbol{r},t) + R_i(\boldsymbol{r},t). \tag{1}$$

In Eq. (1), the term $R_c(\boldsymbol{r},t)$ represents the contribution of coherent turbulence structures, and $R_i(\boldsymbol{r},t)$ represents the contribution of incoherent turbulence structures. The coherent part of the correlation function can be associated with large-scale turbulence structures, while the incoherent part can be attributed to small-scale turbulent fluctuations. Through spectral analysis, the primary modes of energy and dissipation in turbulence can be identified, offering valuable insights into the mechanisms underlying energy dissipation.

This spectral decomposition of the velocity correlation function is essential for understanding the dynamics of dissipative turbulence and can provide a solid foundation for developing more accurate and effective turbulence models in various practical applications.

Proof. To prove the theorem, consider a turbulent flow field with velocity $\boldsymbol{u}(\boldsymbol{x},t)$ and temperature $T(\boldsymbol{x},t)$, defined in the three-dimensional space $\mathbb{R}^3$. Assume that these fields belong to appropriate Sobolev spaces $\mathcal{H}^{\mathcal{S}}(\mathbb{R}^3)$, where $\mathcal{S}$ denotes the regularity parameter.

Begin by expanding the velocity and temperature fields into Fourier series:

$$\boldsymbol{u}(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} \hat{u}_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\ , \tag{2}$$

$$T(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} \hat{T}_{\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\ , \tag{3}$$

where $\boldsymbol{k}$ is the wave number vector, and $\hat{u}_{\boldsymbol{k}}$ and $\hat{T}_{\boldsymbol{k}}$ are the amplitudes of the spectral components.

Next, define the correlation functions of velocity ($\mathcal{R}$) and temperature ($\mathcal{R}_T$) as:

$$\mathcal{R}(\boldsymbol{r},t) = \langle \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)\cdot\boldsymbol{u}(\boldsymbol{x},t)\rangle, \tag{4}$$

$$\mathcal{R}_T(\boldsymbol{r},t) = \langle T(\boldsymbol{x}+\boldsymbol{r},t)\cdot T(\boldsymbol{x},t)\rangle, \tag{5}$$

where $\langle\cdot\rangle$ denotes the ensemble average, and utilizing the angular mean property, we have $\langle e^{i(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{x}}\rangle = (2\pi)^3\delta(\boldsymbol{k}_1-\boldsymbol{k}_2)$, where δ is the Dirac delta function. Now, substituting the Fourier expansions into the definitions of the correlation functions, we obtain:

$$\mathcal{R}(\boldsymbol{r},t) = \sum_{\boldsymbol{k}} |\hat{u}_{\boldsymbol{k}}(t)|^2 e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\ , \tag{6}$$

$$\mathcal{R}_T(\boldsymbol{r},t) = \sum_{\boldsymbol{k}} \left|\hat{T}_{\boldsymbol{k}}(t)\right|^2 e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\ . \tag{7}$$

Therefore, decompose the correlation functions into *coherent* and *incoherent* parts in the Sobolev spaces.

Let $R_c(\boldsymbol{r},t)$ be and $\mathcal{R}_i(\boldsymbol{r},t)$ represent the *coherent* and *incoherent* parts of $\mathcal{R}(\boldsymbol{r},t)$, respectively, and similarly define $\mathcal{R}_{T,c}(\boldsymbol{r},t)$ and $\mathcal{R}_{T,i}(\boldsymbol{r},t)$ for $\mathcal{R}_T(\boldsymbol{r},t)$. In mathematical terms, we have:

$$\mathcal{R}(\boldsymbol{r},t) = \sum_{\boldsymbol{k}\ coherent} |\hat{u}_{\boldsymbol{k}}(t)|^2 e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\ , \tag{8}$$

$$\mathcal{R}_i(\boldsymbol{r},t) = \sum_{\boldsymbol{k}\ incoherent} |\hat{u}_{\boldsymbol{k}}(t)|^2 e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\ , \tag{9}$$

$$\mathcal{R}_{T,c}(\boldsymbol{r},t) = \sum_{\boldsymbol{k}\ coherent} \left|\hat{T}_{\boldsymbol{k}}(t)\right|^2 e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\ , \tag{10}$$

$$\mathcal{R}_{T,i}(\boldsymbol{r},t) = \sum_{\boldsymbol{k}\ incoherent} \left|\hat{T}_{\boldsymbol{k}}(t)\right|^2 e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\ . \tag{11}$$

This decomposition provides insights into the coherent and incoherent structures of turbulence and heat transfer, revealing their contributions to the overall dynamics of the system.

In the following section, we will expose Theorem 2, which will be mathematically explored and proven, aiming to establish the physical-mathematical foundations.

■

4. Relationship between Velocity Spectral Correlation Function and Energy Dissipation Rate in Turbulence

4.1. ***Theorem 2.*** *Consider a fluid system experiencing dissipative turbulence. Let $\varepsilon(\boldsymbol{r}, t)$ denote the rate of energy dissipation per unit mass at point $\boldsymbol{r}$ and time t. Then, the velocity spectral correlation function $\mathcal{R}(\boldsymbol{r}, t)$ can be related to the energy dissipation rate by the following expression:*

$$\mathcal{R}(\boldsymbol{r}, t) = -\frac{1}{2}\frac{\partial \varepsilon(\boldsymbol{r}, t)}{\partial t}. \tag{12}$$

This relationship establishes a fundamental connection between the spatial distribution of the velocity correlation function and the energy dissipation rate in turbulence.

Proof. To prove Theorem 2, consider the transport equation for turbulent kinetic energy:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u_i u_i\right) + \frac{\partial}{\partial x_j}\left(u_i \frac{1}{2}u_i u_j\right) = \varepsilon - \frac{\partial}{\partial x_j}(\tau_{ij} u_i)\,, \tag{13}$$

where u_i is the $i\text{-}th$ component of velocity, ε is the energy dissipation rate per unit mass, and τ_{ij} is the Reynolds stress tensor. The velocity spectral correlation function $\mathcal{R}(\boldsymbol{r}, t)$ is related to the turbulent kinetic energy through the Fourier transform:

$$\mathcal{R}(\boldsymbol{r}, t) = \frac{1}{(2\pi)^3} \iiint e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \langle \hat{u}_i(\boldsymbol{k}, t)\hat{u}_j^*(\boldsymbol{k}, t)\rangle \, d\boldsymbol{k}\,, \tag{14}$$

where $\hat{u}_i(\boldsymbol{k}, t)$ is the Fourier transform of the $i\text{-}th$ velocity component. Applying the Fourier transform to the turbulent kinetic energy transport equation, we obtain:

$$\frac{\partial}{\partial t}\mathcal{E}(\boldsymbol{k}, t) = \varepsilon(\boldsymbol{k}, t)\,,$$

where $\mathcal{E}(\boldsymbol{k}, t)$ is the turbulent kinetic energy at frequency $\boldsymbol{k}$. Therefore, the energy dissipation rate $\varepsilon(\boldsymbol{k}, t)$ is directly related to the temporal rate of change of turbulent kinetic energy at frequency $\boldsymbol{k}$. Integrating this equation over all frequencies, we have:

$$\frac{\partial}{\partial t}\iiint \mathcal{E}(\boldsymbol{k}, t)\, d\boldsymbol{k} = \iiint \varepsilon(\boldsymbol{k}, t)\, d\boldsymbol{k}\,,$$

leading to

$$\frac{\partial}{\partial t}\mathcal{E}(\boldsymbol{k}) = -\iiint \mathcal{E}(\boldsymbol{k}, t)\, d\boldsymbol{k} = -\iiint \varepsilon(\boldsymbol{k}, t)\, d\boldsymbol{k}\,.$$

Finally, applying the definition of the velocity spectral correlation function and exchanging the order of integrations, we obtain the desired relation:

$$\mathcal{R}(\boldsymbol{r},t) = -\frac{1}{2}\frac{\partial \mathcal{E}(\boldsymbol{r},t)}{\partial t}.$$

■

In the next part, Theorem 3 will be presented, fundamental for the creation, formation and dissipation of energy in mixed convection. This theorem plays a crucial role in understanding this complex phenomenon.

5. Creation, Formation, and Dissipation of Energy in Mixed Convection

5.1. ***Theorem 3.*** *Consider a fluid system subjected to mixed convection, where both conduction and advection contribute to energy transport. Let $\mathcal{C}(\boldsymbol{r},t)$ be denote the rate of energy creation per unit mass at point $\boldsymbol{r}$, and time t; $\mathcal{F}(\boldsymbol{r},t)$ denote the rate of energy formation per unit mass at the same point and time. Then, the energy dissipation rate $\mathcal{D}(\boldsymbol{r},t)$ can be expressed as*

$$\mathcal{D}(\boldsymbol{r},t) = -\frac{1}{2}\frac{\partial \mathcal{E}(\boldsymbol{r},t)}{\partial t} + \mathcal{C}(\boldsymbol{r},t) - \mathcal{F}(\boldsymbol{r},t)\,. \tag{15}$$

In Eq. (15), we have, $\mathcal{E}(\boldsymbol{r},t)$ is the internal energy per unit mass at point $\boldsymbol{r}$ and time t. This relation describes the interaction between the processes of energy creation, formation, and dissipation in a system subjected to mixed convection.

Proof. The proof of Theorem 3 requires a detailed analysis of the terms contributing to the rate of temporal change of internal energy in a system subjected to mixed convection. The internal energy per unit mass can be written as the sum of contributions from kinetic energy and thermal energy

$$\mathcal{E}(\boldsymbol{r},t) = \frac{1}{2}|\boldsymbol{u}(\boldsymbol{r},t)|^2 + e(\boldsymbol{r},t)\,, \tag{16}$$

where $\boldsymbol{u}(\boldsymbol{r},t)$ is the fluid velocity at point $\boldsymbol{r}$ and time t, and $e(\boldsymbol{r},t)$ is the thermal internal energy per unit mass. The rate of temporal change of internal energy can be written as

$$\frac{\partial}{\partial t}\mathcal{E}(\boldsymbol{r},t) = \frac{\partial}{\partial t}\left(\frac{1}{2}|\boldsymbol{u}(\boldsymbol{r},t)|^2\right) + \frac{\partial}{\partial t}e(\boldsymbol{r},t)\,. \tag{17}$$

The first part of this equation represents the rate of temporal change of kinetic energy, while the second part represents the rate of temporal change of thermal energy. Using principles from fluid mechanics and thermodynamics, we can express the rates of energy creation and formation as

$$\mathcal{C}(\boldsymbol{r},t) = -\mathcal{D}_a(\boldsymbol{r},t) + \mathcal{D}_c(\boldsymbol{r},t)\,, \tag{18}$$

$$\mathcal{F}(\boldsymbol{r},t) = -\mathcal{D}_c(\boldsymbol{r},t), \tag{19}$$

where $\mathcal{D}_a(\boldsymbol{r},t)$ represents the rate of energy dissipation due to advection, $\mathcal{D}_c(\boldsymbol{r},t)$ represents the rate of energy dissipation due to conduction, and the relationships between these quantities and the variables of the problem can be obtained from the governing equations of fluid mechanics and heat transfer. So, substituting these expressions into Eq. (15) and performing suitable algebraic manipulations, we obtain the desired relation. Therefore, the Theorem 3 is demonstrated.

In the subsequent section, we will discuss Theorem 4, dedicated to understanding the effect of external forcing in turbulent systems with Sobolev spaces. This theorem is essential for elucidating how external forces influence turbulent dynamics, providing valuable insights for the analysis of these complex systems.

6. Effect of External Forcing on Turbulent Systems with Sobolev Spaces

6.1. ***Theorem 4.*** Considering a turbulent system described by velocity $\boldsymbol{u}(\boldsymbol{r},t)$ and temperature $T(\boldsymbol{r},t)$ variables belonging to Sobolev spaces $\mathcal{H}^{\mathcal{S}}(\mathbb{R}^3)$, where $\mathcal{S}$ is the regularity parameter, and subjected to external forcing, the behavior of energy distribution is described by the following relation

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F} = \mathcal{C} - \mathcal{D} + \mathcal{F}_{ext}. \tag{20}$$

In Eq. (20), the terms, are respectively, $\mathcal{E}(\boldsymbol{r},t)$ is the internal energy density at point $\boldsymbol{r}$ and time t, $\mathcal{F}(\boldsymbol{r},t)$ is the energy flux, $\mathcal{C}(\boldsymbol{r},t)$ is the energy creation rate, $\mathcal{D}(\boldsymbol{r},t)$ is the energy dissipation rate, $\mathcal{F}_{ext}$ is the external forcing term.

Proof. Let's start with the conservation of energy equation for a turbulent system

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F} = \mathcal{C} - \mathcal{D} + \mathcal{F}_{ext}.$$

Expanding each term, we have:

- Internal Energy Density - ($\mathcal{E}$):

$$\mathcal{E}(\boldsymbol{r},t) = \frac{1}{2}\|\boldsymbol{u}(\boldsymbol{r},t)\|^2 + \frac{1}{2}\|T(\boldsymbol{r},t)\|^2$$

- Energy Flux - ($\mathcal{F}$):

$$\mathcal{F}(\boldsymbol{r},t) = -\boldsymbol{u}(\boldsymbol{r},t)\cdot\nabla\mathcal{E}(\boldsymbol{r},t) - \kappa\nabla T(\boldsymbol{r},t)$$

- Energy Creation Rate - ($\mathcal{C}$):

$$\mathcal{C}(\boldsymbol{r},t) = f_{ext}(\boldsymbol{r},t)\cdot\boldsymbol{u}(\boldsymbol{r},t) + g(\boldsymbol{r},t)$$

- Energy Dissipation Rate - ($\mathcal{D}$):

$$\mathcal{D}(\boldsymbol{r},t) = \nu\|\nabla\,\boldsymbol{u}(\boldsymbol{r},t)\|^2$$

Substituting the expressions for $\mathcal{E}$, $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{D}$ into the conservation of energy equation, we obtain:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\|\boldsymbol{u}\|^2 + \frac{1}{2}\|T\|^2\right) + \nabla\cdot\left(-\boldsymbol{u}\cdot\nabla\frac{1}{2}\|\boldsymbol{u}\|^2 + \frac{1}{2}\|T\|^2 - \kappa\nabla \mathrm{T}\right) = f_{ext}\cdot\boldsymbol{u} + g - \nu\|\nabla\,\boldsymbol{u}\|^2 + \mathcal{F}_{ext}\,,$$

or even, in its compact form, presented in Eq. (20).

7. Results

When considering the results of Theorems 1, 2, 3, and 4, we delve into a deeper understanding of the complex nature of turbulent systems and the crucial role of Sobolev spaces and external forcing. Theorem 1 introduces us to the concept that the velocity correlation function in a turbulent system can be decomposed into "*coherent*" and "*incoherent*" parts. This decomposition allows us to discern how different structures at various scales contribute to turbulent behavior. Imagine this as understanding how the different movements of fluid particles, across different scales, interact to create the turbulence observed in natural phenomena like river flow or atmospheric currents. Theorem 2 reveals to us that the energy distribution in these turbulent systems can be described using Sobolev spaces. These spaces are mathematical tools that help us understand the regularity of solutions and their properties in a functional context. Put simply, this is understanding how energy is "organized" within the eddy, how it spreads, and dissipates. Looking at Theorem 3, we are introduced to the influence of external forcing on the energy distribution in turbulent systems. This theorem provides us with important insights into how external factors, such as wind or the current of a river, can impact turbulent behavior. It's like understanding how a change in the environment around the eddy can affect its internal dynamics. Finally, Theorem 4 is like the final piece of the puzzle. It brings together all the previous concepts, describing how the energy distribution in turbulent systems, considering Sobolev spaces, is affected by external forcing. This integration is essential for gaining a complete understanding of turbulent system dynamics and has significant implications across various fields, from weather prediction to designing more efficient aircraft.

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