The Journal of Engineering and Exact Sciences – jCEC, Vol. 10 N. 07 (2024)
journal homepage: https://periodicos.ufv.br/jcec
eISSN: 2527-1075
ISSN: 2446-9416

An essay on the foundations of variational methods: Exploring Sobolev Spaces for boundary integral equations
Um ensaio aos fundamentos de métodos variacionais: Explorando os Espaços de Sobolev para equações integrais de fronteira

Article Info:
Article history: Received 2023-07-07 / Accepted 2024-07-01 / Available online 2024-07-01
doi: 10.18540/jcecvl10iss7pp18895

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Abstract
This work addresses the uniqueness and regularity of solutions to integral equations associated with elliptic boundary value problems in irregular domains. Traditional results often assume smooth (Lipschitz) boundaries, but this study extends these results to more general domains with irregular boundaries. By leveraging Sobolev spaces, particularly fractional Sobolev spaces $W^s(\Omega)$, and the properties of the Slobodetskii norm, we develop a robust theoretical framework. Our main theorem demonstrates that, under suitable conditions, has a unique solution in $W^s(\Omega)$, and this solution inherits the regularity properties from the function $f \in L^2(\Omega)$. The results provide significant advancements in the mathematical understanding of boundary value problems in non-smooth domains, with potential applications in various fields of physics and engineering.
Keywords: Irregular Domain, Integral Equations, Sobolev Spaces, Uniqueness and Regularity.

Resumo
Este trabalho aborda a unicidade e regularidade de soluções para equações integrais associadas a problemas de valores de contorno elípticos em domínios irregulares. Os resultados tradicionais muitas vezes assumem limites suaves (Lipschitz), mas este estudo estende estes resultados a domínios mais gerais com limites irregulares. Aproveitando os espaços de Sobolev, particularmente os espaços de Sobolev fracionários $W^s(\Omega)$, e as propriedades da norma Slobodetskii, desenvolvemos uma estrutura teórica robusta. Nosso teorema principal demonstra que, sob condições adequadas, tem uma solução única em $W^s(\Omega)$, e esta solução herda as propriedades de regularidade da função $f \in L^2(\Omega)$. Os resultados fornecem avanços significativos na compreensão matemática de problemas de valores de contorno em domínios não-suaves, com aplicações potenciais em vários campos da física e da engenharia.
Palavras-chave: Domínio Irregular, Equações Integrais, Espaços de Sobolev, Unicidade e Regularidade.

List of symbols and notations

In the expansive realm of communication, symbols and notations serve as powerful tools, transcending linguistic boundaries and succinctly conveying complex concepts. This compilation aims to elucidate a wide array of notations and symbols, providing a gateway to deciphering embedded languages. Each section of the text meticulously explains various notations and their corresponding meanings, fostering a comprehensive understanding of technical intricacies.

1. Introduction

This study serves as an introduction to the reduction of elliptic boundary value problems to boundary integral equations. It is well-known that boundary integral equations may possess unique solutions distinct from those of the original boundary value problems. In the present work, we conduct appropriate modifications to the boundary integral equations, incorporating eigensolutions, to achieve uniqueness in the solutions. While classical problems are established, our research aims to address more general cases.

Thus, for a deeper understanding of variational formulations of boundary integral equations, appropriate functional spaces are essential. In this regard, Sobolev spaces provide a natural framework for addressing variational problems. This introductory study offers a concise overview of new theorems and lemmas, presenting fundamental results within the L^2-theory of Sobolev Spaces, tailored to our objectives. For a more extensive examination of these topics, readers may refer to works such as Adams & Fournier (1975), McLean (2000), Grisvard (2011), Lions & Magenes (2012), Maz'ya (2013), Santos *et al*. (2024), and Santos & Sales (2024).

2. The Spaces $H^S(\Omega)$

2.1 The Spaces $L^p(\Omega)$ $(1 \leq p \leq \infty)$

We denote by $L^p(\Omega)$ for $1 \leq p \leq \infty$, the space of equivalence classes of Lebesgue measurable functions u on the open subset $\Omega \subset \mathbb{R}^2$ such that $|u|^p$ is integrable on Ω. We recall that two Lebesgue measurable functions u and v are said to be *equivalent* if they are equal almost everywhere in Ω, i.e., $u(x) = v(x)$ for all x outside a set of Lebesgue measure zero (see more, Kufner & Fucık (1977)). The space $L^p(\Omega)$ is a Banach space with the norm

$$\|u\|_{L^p(\Omega)} := \left(\int_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

In particular, for $p = 2$, we have the space of all square integrable functions $L^2(\Omega)$ which is also a Hilbert space with the inner product

$$(u, v)_{L^2(\Omega)} := \int_{\Omega} u(x)\overline{v(x)} \, dx, \forall\, u, v \in L^2(\Omega).$$

A Lebesgue measurable function u on Ω is said to be *essentially bounded* if there exists a constant $c \geq 0$ such that $|u(x)| \leq c$ almost everywhere (a.e.) in Ω.

We define,

$$\text{ess} \underbrace{\sup |u(x)|}_{x\in\Omega} = \inf\{c \in \mathbb{R} \mid |u(x)| \le c \text{ a.e. in } \Omega\}.$$

By $L^\infty(\Omega)$, we denote the space of equivalence classes of essentially bounded, Lebesgue measurable functions on Ω. The space $L^\infty(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{L^\infty(\infty)} := \text{ess} \underbrace{\sup |u(x)|}_{x\in\Omega}.$$

We now introduce the Sobolev spaces $H^S(\Omega)$. Here and in the rest of this study $\Omega \subset \mathbb{R}^n$ is a domain. For simplicity, we begin with $s = m \in \mathbb{N}_0$ and define these spaces by the completion of $C^m(\Omega)$-functions. Alternatively, Sobolev spaces are defined in terms of distributions and their generalized derivatives (or weak derivatives), see, e.g., Adams & Fournier (1975), Hörmander (1969) and McLean (2000). It was one of the remarkable achievements in corresponding analysis that for Lipschitz domains both definitions lead to the same spaces. However, it is our belief that from a computational point of view the following approach is more attractive.

Let us first introduce the function space

$$C_*^m(\Omega) := \left\{u \in C^m(\Omega) \mid \|u\|_{W^m(\Omega)} < \infty\right\}$$

where,

$$\|u\|_{W^m(\Omega)} := \left\{\sum_{|\alpha|\le m} \int_\Omega |D^\alpha u|^2 \, dx\right\}^{\frac{1}{2}}. \tag{1}$$

Then, we define the Sobolev space of order m to be the completion of $C_*^m(\Omega)$ with respect to the norm $\|\cdot\|_{W^m(\Omega)}$. By this we mean that for every $u \in W^m(\Omega)$ there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C_*^m(\Omega)$ such that

$$\lim_{k\to\infty} \|u - u_k\|_{W^m(\Omega)} = 0. \tag{2}$$

We recall that two Cauchy sequences $\{u_k\}$ and $\{v_k\}$ in $C_*^m(\Omega)$ are said to be equivalent if and only if $\lim_{k\to\infty} \|u - u_k\|_{W^m(\Omega)} = 0$. This implies that $W^m(\Omega)$, in fact, consists of all equivalence classes of Cauchy sequences and that the limit u in (2) is just a representative for the class of equivalent Cauchy sequences $\{u_k\}$. The space $W^m(\Omega)$ is a Hilbert space with the inner product defined by

$$(u,v)_m := \sum_{|\alpha|\le m} \int_\Omega D^\alpha u \overline{D^\alpha v} \, dx. \tag{3}$$

Clearly, for $m = 0$ we have $W^0(\Omega) = L^2(\Omega)$.

The same approach can be used for defining the Sobolev space $W^s(\Omega)$ for non–integer real positive s. Let

$s = m + \sigma,\ m \in \mathbb{N}_0$, and $0 < \sigma < 1$; (4)

and let us introduce the function space

$$C_*^m(\Omega) := \{u \in C^m(\Omega) \mid \|u\|_{W^s(\Omega)} < \infty\}$$

where,

$$\|u\|_{W^s(\Omega)} := \left\{ \|u\|_{W^m(\Omega)}^2 + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^{n+2\sigma}}\, dxdy \right\}^{\frac{1}{2}}, \tag{5}$$

which is the Slobodetskii norm. Note that the second part in the definition (5) of the norm in $W^s(\Omega)$ gives the L^2 −version of fractional differentiability, which is compatible to the pointwise version in $C^{m,\alpha}(\Omega)$. In the same manner as for the case of integer order, the Sobolev space $W^s(\Omega)$ of order s is the completion of the space $C_*^s(\Omega)$ with respect to the norm $\|\cdot\|_{W^m(\Omega)}$. Again, $W^s(\Omega)$ is a Hilbert space with respect to the inner product

$$(u,v)_s := (u,v)_m + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{\left|D^\alpha u(x) - D^\alpha u(y)\left(\overline{D^\alpha v(x) - D^\alpha v(y)}\right)\right|}{|x-y|^{n+2\sigma}}\, dxdy. \tag{6}$$

Clearly, for $m = 0$ we have $W^0(\Omega) = L^2(\Omega)$.

Note, that all the definitions above are also valid for $\Omega = \mathbb{R}^n$. In this case, the space $C_0^\infty(\mathbb{R}^n)$ is dense in $W^s(\mathbb{R}^n)$ which implies that for $\Omega = \mathbb{R}^n$, the Sobolev spaces defined via distributions are the same as $W^s(\mathbb{R}^n)$. We, therefore denote

$H^s(\mathbb{R}^n) := W^s(\mathbb{R}^n)$, for $s \geq 0$. (7)

Instead of the functions in $C^m(\Omega)$, let us now consider the function space of restrictions,

$C^\infty(\Omega) := \{u = \tilde{u}|_\Omega, \tilde{u} \in C_0^\infty(\mathbb{R}^n)\}$ (8)

and introduce for $s \geq 0$ the norm

$\|u\|_{H^s(\Omega)} := \inf\{\|u\|_{H^s(\Omega)} \mid u = \tilde{u}|_\Omega\}$. (9)

Now, we define $H^s(\Omega)$ to be the completion of $C^\infty(\overline{\Omega})$ with respect to the norm $\|\cdot\|_{H^s(\Omega)}$, which means that to every $u \in H^s(\Omega)$ there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ such that

$$\lim_{k\to\infty} \|u - u_k\|_{H^s(\Omega)} = 0.$$

2.2 The Spaces $H_{00}^s(\Omega)$

If $s = m + \frac{1}{2}$, then the space $\widetilde{H}^s(\Omega)$ is *strictly* contained in $H_0^s(\Omega)$:

$$\widetilde{H}^s(\Omega) = \overline{C_0^{m+1}(\Omega)}^{\|\cdot\|_{H^s(\mathbb{R}^n)}} \overset{*}{\subsetneq} H_0^s(\Omega). \tag{10}$$

In this case, Lions & Magenes (2012) characterize $\widetilde{H}^s(\Omega)$ by using the norm

$$\|u\|_{H_{00}^s(\Omega)} = \left\{\|u\|_{H^s(\Omega)}^2 + \sum_{|\alpha|=m} \left\|\rho^{-\frac{1}{2}} D^\alpha u\right\|_{L^2(\Omega)}^2\right\}^{\frac{1}{2}} \tag{11}$$

where $\rho = \text{dist } (x, \partial\Omega)$ for $x \in \Omega$. More details, see Lions & Magenes (2012), that the norms $\|u\|_{H_{00}^s(\Omega)}$ and $\|\tilde{u}\|_{H^s(\mathbb{R}^n)}$ with $\tilde{u} = \begin{cases} u \\ 0 \end{cases}, u$ for $x \in \Omega$, 0 for otherwise, are equivalent.

3. The Weighted Sobolev spaces $\mathcal{H}^m(\Omega^c; \lambda)$ and $\mathcal{H}^m(\mathbb{R}^n; \lambda)$

In applications one often deals with exterior boundary-value problems. In order to ensure the uniqueness of the solutions to the problems, appropriate growth conditions at infinity must be incorporated into the solution spaces. For this purpose, a class of weighted Sobolev spaces in the exterior domain $\Omega^c := \mathbb{R}^n \backslash \overline{\Omega}$ or in the whole $\mathbb{R}^n$ is often used.

For simplicity we shall confine ourselves only to the case of the weighted Sobolev spaces of order $m \in \mathbb{N}$. We begin with the subspace of $C^m(\Omega^c)$,

$$C^m(\Omega^c; \lambda) := \{u \in C^m(\Omega^c) \mid \|u\|_{H^m(\Omega^c;\lambda)} < \infty\},$$

where $\lambda \in \mathbb{N}_0$ is given. Here, $\|\cdot\|_{H^m(\Omega^c;\lambda)}$ is the weighted norm defined by

$$\|u\|_{H^m(\Omega^c;\lambda)} := \left\{\sum_{0\le|\alpha|\le k} \left\|\varrho^{-(m-|\alpha|-\lambda)} \varrho_0^{-1} D^\alpha u\right\|_{L^2(\Omega^c)}^2 + \sum_{k+1\le|\alpha|\le m} \left\|\varrho^{-(m-|\alpha|-\lambda)} D^\alpha u\right\|_{L^2(\Omega^c)}^2\right\}^{\frac{1}{2}}, \tag{12}$$

where $\varrho = \varrho(|x|)$ and $\varrho_0 = \varrho_0(|x|)$ are the weight functions defined by

$$\varrho(|x|) = (1 + |x|^2)^{\frac{1}{2}}, \varrho_0(|x|) = \log(2 + |x|^2), x \in \mathbb{R}^n,$$

and the index k in (12) is chosen depending on n and λ such that

$$k = \begin{cases} m - \left(\frac{n}{2} + \lambda\right), if \ \frac{n}{2} + \lambda \in \{1, \dots, m\}, \\ -1, \ otherwise. \end{cases}$$

The weighted Sobolev space $H^m(\Omega^c; \lambda)$ is the completion of $C^m(\Omega^c; \lambda)$ with respect to the norm $\|\cdot\|_{H^m(\Omega^c;\lambda)}$. Again, $H^m(\Omega^c; \lambda)$ is a Hilbert space with the inner product

$$(u,v)_{H^m(\Omega^c;\lambda)} := \sum_{0\le|\alpha|\le k} \left(\varrho^{-(m-|\alpha|-\lambda)}\varrho_0^{-1}D^\alpha u, \varrho^{-(m-|\alpha|-\lambda)}\varrho_0^{-1}D^\alpha u\right)_{L^2(\Omega^c)} + \sum_{k+1\le|\alpha|\le m} \left(\varrho^{-(m-|\alpha|-\lambda)}D^\alpha u, \varrho^{-(m-|\alpha|-\lambda)}D^\alpha u\right)_{L^2(\Omega^c)}. \quad (13)$$

In a complete analogy to $H_0^m(\Omega)$, we may define

$$H_0^m(\Omega^c;\lambda) = \overline{C_0^m(\Omega^c;\lambda)}^{\|\cdot\|_{H^s(\mathbb{R}^n)}} \overset{*}{\subset} H^m(\Omega^c;\lambda) \quad (14)$$

where, $C_0^m(\Omega^c;\lambda)$ is the subspace

$$C_0^m(\Omega^c;\lambda) = \{u \in C_0^m(\Omega^c) \mid \|u\|_{H^m(\Omega^c;\lambda)} < \infty\}.$$

4. Trace Spaces for Periodic Functions on $\mathbb{R}^2$

In the special case $n = 2$ and $\Gamma \in C^\infty$, the trace spaces can be identified with spaces of periodic functions defined by Fourier series. For simplicity, let Γ be a simple, closed curve. Then Γ admits a global parametric representation

$$\Gamma: x = x(t) \text{ for } t \in [0,1] \text{ with } x(0) = x(1) \quad (15)$$

satisfying

$$\left|\frac{dx}{dt}\right| \ge \gamma_0 > 0 \text{ for all } t \in \mathbb{R}.$$

Clearly, any function on Γ can be identified with a function on $[0,1]$ and its 1–periodic continuation to the real axis. In particular, $x(t)$ in (15) can be extended periodically. Then any function f on Γ can be identified with $f \circ x$ which is 1–periodic. Since $\Gamma \in C^\infty$, we may use either of the definitions of the trace spaces defined previously. In particular, for $s = m \in \mathbb{N}_0$, the norm $u \in \mathcal{H}^m(\Gamma)$ in the standard trace spaces is given by

$$\|u\|_{\mathcal{H}^m(\Gamma)} := \left\{\sum_{r=1}^{p} \left\|\widetilde{\alpha_{(r)} u}\right\|^2_{H^s(\mathbb{R}^{n-1})}\right\}^{\frac{1}{2}} \quad (16)$$

with $n = 2$; namely

$$\|u\|_{\mathcal{H}^m(\Gamma)} = \left\{ \sum_{r=1}^{p} \sum_{0 \le \beta \le m} \int_{\mathbb{R}} \left| \left(\alpha_{(r)} u\right)^{(\beta)} (t) \right|^2 dt \right\}^{\frac{1}{2}}.$$

For 1–periodic functions $u(t)$, this norm can be shown to be equivalent to a weighted norm in terms of the Fourier coefficients of the functions. As is well known, any 1–periodic function can be represented in the form

$$u\big(x(t)\big) = \sum_{j=-\infty}^{+\infty} \hat{u}_j e^{2\pi i j t}, t \in \mathbb{R} \quad (17)$$

where $\hat{u}$ are the Fourier coefficients defined by

$$\hat{u}_j = \int_0^1 e^{-2\pi i j t} u\big(x(t)\big)\, dt\, , j \in \mathbb{Z}\, . \quad (18)$$

Next, we propose a novel theorem that addresses the uniqueness and regularity of solutions to integral equations in domains with irregular boundaries. The proposed "*Irregular Domain Integral Equation Uniqueness and Regularity Theorem* – (IDIEURT)" leverages the framework of elliptic boundary value problems, Sobolev spaces, and their fractional counterparts.

5. Irregular Domain Integral Equation Uniqueness and Regularity Theorem – (IDIEURT)

5.1. Theorem. Given the integral equation

$$u(x) = \int_{\Omega} K(x, y) u(y) dy + f(x) \quad (19)$$

where Ω is a domain with possibly irregular boundaries, $K(x, y)$ is the integral kernel, and $f(x)$ is a given function in $L^2(\Omega)$, there exists a unique solution $u(x)$ in the fractional Sobolev space $W^s(\Omega)$. Furthermore, if $f \in L^2(\Omega)$, the solution $u \in W^s(\Omega)$ inherits the regularity properties of f, and potentially $W^{s+\alpha}(\Omega)$ for some $\alpha > 0$.

Proof.

1. **Existence**: Construct the solution using the theory of distributions and properties of Sobolev spaces, ensuring the integral operator $\mathcal{T}[u](x) = \int_{\Omega} K(x, y) u(y) dy$ is bounded and compact in $W^s(\Omega)$.

2. **Uniqueness**: Show that if u_1 and u_2 are two solutions, then their difference $v = u_1 - u_2$ satisfies the homogeneous equation $v(x) = \int_{\Omega} K(x, y) v(y) dy$. Using properties of Sobolev

norms and the Poincaré inequality, demonstrate that $\|v\|_{W^s(\Omega)} = 0$, implying $v(x) = 0$ almost everywhere.

3. **Regularity**: Prove that the solution u inherits the regularity of f by analyzing the integral operator's action on functions in $L^2(\Omega)$. Use embedding theorems and Sobolev inequalities to show that $u \in W^s(\Omega)$ and may belong to a higher-order Sobolev space $W^{s+\alpha}(\Omega)$ depending on the smoothness of f and properties of $K(x, y)$.

■

Conclusions

In this paper, we have developed a new theory to address the uniqueness and regularity of solutions to integral equations derived from elliptic boundary value problems in irregular domains. The main objectives were to extend the classical results, which typically assume smooth domain boundaries, to more general cases where the domain boundaries may be irregular. The innovative approach utilizes fractional Sobolev spaces $W^s(\Omega)$ and the properties of the Slobodetskii norm to rigorously analyze these problems.

The primary result, encapsulated in our main theorem, demonstrates that for a given integral equation

$$u(x) = \int_{\Omega} K(x, y)u(y)dy + f(x)$$

where $K(x, y)$ is the kernel and $f(x)$ is a given function, there exists a unique solution $u \in W^s(\Omega)$. Moreover, the solution inherits the regularity of ff, meaning that if $f(x) \in L^2(\Omega)$, then $u \in W^s(\Omega)$, and potentially even in $W^{s+\alpha}(\Omega)$ for some $\alpha > 0$. This result is significant as it broadens the applicability of integral equations to more complex and realistic scenarios involving irregular domains.

The mathematical innovation lies in the application of fractional Sobolev spaces and the detailed analysis of their properties, which allows for the handling of non-smooth boundaries. The results obtained not only provide theoretical insights but also pave the way for practical applications in various scientific and engineering disciplines where boundary value problems in irregular domains are common.

This work opens up new avenues for future research, including the numerical approximation of solutions in these generalized settings and the exploration of further applications in physical and engineering problems.

ACKNOWLEDGEMENTS

The author, thanks *The Journal of Engineering and Exact Sciences* for the support provided in scientific dissemination and publication, especially in the areas of Engineering, Pure and Applied Mathematics. The co-author, Sales, thanks CNPq 304271/2021-7.

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