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Fixed Points Theorems of Multi-Valued Mappings in $\mathcal{F}$-Metric spaces

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Abstract
We prove existence fixed point results of generalized multi-valued g- weak contraction mappings and multivalued mappings satisfying a Reich-type condition in $\mathcal{F}$- metric spaces. Our results generalized, extend and enrich recently fixed point existing in the literature. Examples and applications illustrating the main resuts are presented in the last section.
Keywords: Fixed point, $\mathcal{F}$- metric space, $\mathcal{F}$- Hausdorff distance, Multi-valued mapping.

1. Introduction and preliminaries

Recently, Jleli and Samet have introduced a new concept named $\mathcal{F}$ -metric spaces as a generalization of the notion of the metric spaces [3]. The main objective of the present paper is to prove the common fixed point theorems for generalized multi-valued g - weak contraction mappings in $\mathcal{F}$ -metric spaces, and which presents a generalization of some previous theories such as [1], [4,5] and [7], which have been used in $\mathcal{F}$ -metric spaces. It is worthy to mention that the obtained results will allow generalizing and unifying Nadler's multi-valued contraction mapping and many fixed point theorems for multivalued mappings. In $\mathcal{F}$-metric spaces. Furthermore, this paper will present some applications and examples to validate the proposed theorems.

Firstly, a brief relocation of basic notions and facts on $\mathcal{F}$-metric spaces are exposed. Let's denote by $\mathcal{F}$ the set of functions $f:]0,\infty[\to \mathbb{R}$ such that

$(\mathcal{F}_1)$ f is non-decreasing, i.e., $0 < s < t$ implies $f(s) \leq f(t)$.

$(\mathcal{F}_2)$ For every sequence $(t_n) \subset]0,\infty[$, we have

$$\lim_{n\to+\infty} t_n = 0 \; if \; and \; only \; \lim_{n\to+\infty} f(t_n) = -\infty.$$

Definition 1.1 ([3, Definition 2.1]) Let E be a nonempty set and $D: E^2 \to \mathbb{R}_+$ be a given mapping. Suppose that there exists $(f,a) \in [0,\infty[$, such that

(D_1) $\forall (x,y) \in E^2,\ D(x,y) = 0 \; if \; and \; only \; x = y.$

(D_2) $\forall (x,y) \in E^2,\ D(x,y) = D(y,x).$

(D_3) $\forall (x,y) \in E^2$ and for every $N \in \mathbb{N}, N \geq 2$ and for all $(v_i)_{i=1}^N \subset E$ with $(v_1, v_N) = (x,y)$,

we have

$$D(x,y) > 0 \text{ implies } f\big(D(x,y)\big) \le f(\sum_{i=1}^{N-1} D(v_i, v_{i+1})) + a$$

Then D is called an $\mathcal{F}$-metric on E and the pair (E, D) is called an $\mathcal{F}$ -metric space.

Definition 1.2 ([8, Definition 1.3]) Let Φ be the family of functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

1) ϕ is non-decreasing.
2) The series $\sum_{k=0}^{\infty} \phi^n(t)$ converges for any $t > 0$, where ϕ^n is the n-th iterate of ϕ.

Lemma 1.1 ([8, Lemma 1.4]) Let $\phi \in \Phi$, we have $\phi(t) < t$ for all $t > 0$.

Remark 1.1 If $\phi \in \Phi$, then $\phi(0) = 0$.

If $\phi(0) > 0$, by Lemma 1.1, we have $\phi\big(\phi(0)\big) < \phi(0)$. Since ϕ is non-decreasing, then $\phi(0) \le \phi\big(\phi(0)\big)$, which is a contradiction. Hence $\phi(0) = 0$.

Definition 1.3 ([2, Definition 5]) Let (E, D) be an $\mathcal{F}$-metric space. Define:

$$D(x, A) = inf_{y \in A} D(x, y)$$

and

$$L(A, B) = sup_{x \in A} D(x, B)$$

where $x \in E$ and $A, B \in P(E)$.

Definition 1.4 ([2, Definition 6]) Let (E, D) be an $\mathcal{F}$ -metric space and let M_F be the set of all nonempty $\mathcal{F}$-closed and bounded subsets of E. The $\mathcal{F}$ -Hausdorff distance is defined by:

$$\Delta(A, B) = \max\big(L(A, B), L(B, A)\big) \qquad (\mathbf{1.1})$$

Proposition 1.1 ([2, Proposition 3]) Let (E, D) be an $\mathcal{F}$-metric space with continuous function $f \in \mathcal{F}$and $a \ge 0$. Then (M_F, Δ) is an $\mathcal{F}$-metric space.

Lemma 1.2 ([1, Lemma 1 in $\mathcal{F}$-metric space]) Let (E, D) be a $\mathcal{F}$-metric space with continuous $f \in \mathcal{F}$ and $a \ge 0$. Let $A, B \in M_F$ and $q \in \mathbb{R}, q > 1$ be given. Then, for every $a \in A$ there exists $b \in B$ such that

$$D(a, b) \le q\Delta(A, B) \qquad (\mathbf{1.2})$$

Proof Let $a \in A$ be, if $\Delta(A, B) = 0$ then $a \in B$ and $(\mathbf{1.2})$ holds for $b = a$.

If $\Delta(A, B) > 0$, choose $\epsilon = (q - 1)\Delta(A, B)$, there exists $b \in B$ such that

$$D(a, b) \le D(a, B) + (q - 1)\Delta(A, B)$$

$$\le \Delta(A, B) + (q - 1)\Delta(A, B) = q\Delta(A, B)$$

Remark 1.2 If $f \in \mathcal{F}$ is continuous and satisfies $(\mathcal{F}_1)$ then

$$f\big(inf(A)\big) = inf\big(f(A)\big) \ for\ all\ A \subset \mathbb{R}_+ \ with\ inf(A) > 0.$$

Definition 1.5 ([9] and [10, Definition 2.2]) Let g be a self-map on $\mathcal{F}$-metric space (E, D) and let $T: E \to P(E)$ be a multi-valued mapping.

1) A point $x \in E$ is a fixed point of g (resp.T) if $gx = x$(resp.$x \in Tx$) and the set of fixed points of g (resp. T) is denoted by $F(g)$ (resp. $F(T)$).
2) A point $x \in E$ is a coincidence point of g and T if $g(x) \in Tx$) and the set of coincidence points of g and T is denoted by $C(g, T)$.
3) A point $x \in E$ is a common fixed point of g and T if $x = g(x) \in Tx$ and the set of

common fixed points of g and T is denoted by $F(g,T)$.

2-Main Results

Lemma 2.1 Let $(y_n)_n$ be a sequence in a $\mathcal{F}$-metric space (E,D), such that

$$D(y_{n+1},y_n) \leq \phi\big(D(y_n,y_{n-1})\big) \ for\ all\ n \in \mathbb{N}, \qquad (\mathbf{2.1})$$

where $\phi \in \Phi$. Then $(\mathrm{y_n})_\mathrm{n}$ is an $\mathcal{F}$- Cauchy sequence.

Proof If $D(y_1,y_0)=0$, then

$$D(y_2,y_1) \leq \phi\big(D(y_1,y_0)\big) = \phi(0) = 0, \qquad so\ \ y_2 = y_1 = y_0.$$

We conclude that $y_n = y_0$ for all $n \in \mathbb{N}$, so $(y_n)_n$ is $\mathcal{F}$- Cauchy sequence. Now, we assume $\mathrm{D}(y_1,y_0) > 0$. In condition $(\mathbf{2.1})$ and ϕ is non-decreasing, we have

$$D(y_{n+1},y_n) \leq \phi\big(D(y_n,y_{n-1})\big) \leq \phi^2\big(D(y_{n-1},y_{n-2})\big)$$
$$. \leq \phi^n\big(D(y_1,y_0)\big).$$

So,

$$D(y_{n+1},y_n) \leq \phi^n\big(D(y_1,y_0)\big),\ for\ all\ \ n \in \mathbb{N}. \qquad (\mathbf{2.2})$$

By (D_3) and $(\mathbf{2.2})$, for $m > n$ such that $y_n \neq y_m$, we have

$$f\big(D(y_n,y_m)\big) \leq f\left(\sum_{k=n}^{m-1} D(y_k,y_{k+1})\right) + a \leq f\left(\sum_{k=n}^{m-1} \phi^k\big(D(y_0,y_1)\big)\right) + a.$$

Denote

$$S_n = \sum_{k=0}^{n} \phi^k\big(D(y_0,y_1)\big), \quad n \in \mathbb{N}.$$

Then

$$f\big(D(y_n,y_m)\big) \leq f(S_{m-1} - S_{n-1}) + a \qquad (\mathbf{2.3})$$

Since $\phi \in \Phi$, we have

$$\sum_{k=0}^{\infty} \phi^k\big(D(y_0,y_1)\big) < \infty.$$

It follows that, (S_n) is a convergent sequence. This yields that (S_n) is a Cauchy sequence in$\mathbb{R}$.

By $(\mathcal{F}_2)$ and $(\mathbf{2.3})$, it follows that, $\lim_{n,m\to\infty}(S_{m-1} - S_{n-1}) = 0$, implies

$$\lim_{n,m\to\infty}\big(f(S_{m-1} - S_{n-1}) + a\big) = -\infty,$$

then $\lim_{n,m\to\infty} f\big(D(y_n, y_m)\big) = -\infty$. So $\lim_{n,m\to\infty} D(y_n, y_m) = 0$.

Remark 2.1 ([6, Lemma 1]) Let $(y_n)_n$ be a sequence in a $\mathcal{F}$-metric space (E, D), such that

$$\mathrm{D}(y_{n+1}, y_n) \leq \lambda \mathrm{D}(y_n, y_{n-1}),\ \text{for all } \mathrm{n} \in \mathbb{N},\ \lambda \in \mathbb{R},\ 0 < \lambda < 1 \qquad (\mathbf{2.4})$$

Then $(\mathrm{y_n})_\mathrm{n}$ is an $\mathcal{F}$-Cauchy sequence. Putting $\phi(t) = \lambda t$, where $\lambda \in]0,1[$, we get $\phi \in \Phi$.

Theorem 2.1 Let g be a self-map on $\mathcal{F}$-metric space (E, D) with continuous $f \in \mathcal{F}$ and $a \geq 0$ and let $T: E \to M_F$ be a multi-valued mapping such that

$$\Delta(Tx, Ty) \leq \phi\big(D(gx, gy)\big). \qquad \mathbf{(2.5)}$$

For all $x, y \in E$, where $\phi \in \Phi$ and $Tx \subset g(E)$ for all $x \in E$. Suppose that the following assertions hold:

a- For each $x \in E$ the set

$$E_T(x) = \big\{y \in Tx;\ L(Tx, gx) \leq q\big(\mathrm{D}(y, gx)\big)\ for\ some\ q > 1\big\}$$

is nonempty.

b- $g(E)$ is a $\mathcal{F}$ -complete subspace of E. Then

1) The set $C(g, T)$ is nonempty.
2) If $ggx = gx$ for some $x \in C(g, T)$ then g and T have a common fixed point.

Proof

1) Let $x_0 \in E$ be arbitrary and $y_0 = gx_0$. Since $Tx_0 \subset g(E)$, there exists $x_1 \in E$, such that $y_1 = gx_1 \in Tx_0$. If $\Delta(Tx_0, Tx_1) = 0$, so

$$gx_1 \in Tx_0 = Tx_1.$$

If $\Delta(Tx_0, Tx_1) > 0$. Since $E_T(x_1)$ is nonempty, there exists $y_2 \in Tx_1$ such that

$$L(Tx_1, gx_1) \leq qD(y_2, gx_1)\ \text{ for some } q > 1.$$

Then

$$D(y_2, gx_1) \leq L(Tx_1, gx_1).$$

Since $y_2 \in Tx_1 \subset g(E)$, there exists $x_2 \in E$, such that $y_2 = gx_2 \in Tx_1$. Then

$$D(gx_2, gx_1) \leq L(Tx_1, gx_1) \leq \Delta(Tx_1, Tx_0) \leq \phi\big(D(gx_1, gx_0)\big).$$

We continue with the same process. If $\Delta(Tx_1, Tx_2) = 0$, so

$$gx_2 \in Tx_1 = Tx_2.$$

Now, if $\Delta(Tx_1, Tx_2) > 0$. Since $E_T(x_2)$ is nonempty, there exists $y_3 \in Tx_2$ such that

$$L(Tx_2, gx_2) \leq qD(y_3, gx_2)\ \text{ for some } q > 1.$$

Then

$$D(y_3, gx_2) \leq L(Tx_2, gx_2)$$

Since $y_3 \in Tx_2 \subset g(E)$, there exists $x_3 \in E$, such that $y_3 = gx_3 \in Tx_2$. Then

$$D(gx_3, gx_2) \leq L(Tx_2, gx_2) \leq \Delta(Tx_2, Tx_1) \leq \phi\big(D(gx_2, gx_1)\big).$$

Continuing in this fashion, we produce a sequence $(y_n)_n$ of points of E such that $y_{n+1} = gx_{n+1} \in Tx_n$ and

$$D(y_{n+1}, y_n) \leq L(Tx_n, gx_n) \leq \Delta(Tx_n, Tx_{n-1}) \leq \phi\big(D(y_n, y_{n-1})\big). \forall n \in \mathrm{N}^*.$$

By Lemma 2.1, it follows that $(gx_n)_n$ is a $\mathcal{F}$-Cauchy sequence in a complete $\mathcal{F}$-metric space $(g(E), D)$, hence there exists $x \in E$ such that

$$\lim_{n\to\infty} gx_n = gx.$$

We show that $gx \in Tx$. If $gx \notin Tx$, since Tx is closed, this implies $D(gx, Tx) > 0$. In condition **(2.5)** and by Remark 1.2, we have

$$\begin{aligned} f\big(D(gx, Tx)\big) &\leq f\big(D(gx, gx_{n+1}) + D(gx_{n+1}, Tx)\big) + a \\ &\leq f\big(D(gx, gx_{n+1}) + \Delta(Tx_n, Tx)\big) + a. \\ &\leq f\Big(D(gx, gx_{n+1}) + \phi\big(D(gx_n, gx)\big)\Big) + a \\ &\leq f\big(D(gx, gx_{n+1}) + D(gx_n, gx)\big) + a \end{aligned}$$

Taking limit as $n \to +\infty$, we get $f\big(D(gx, Tx)\big) \leq -\infty$, which is a contradiction, hence $D(gx, Tx) = 0$. Since Tx is closed, then $gx \in Tx$.

2) If $ggx = gx$, for some $x \in C(g, T)$, In condition **(2.5)**, we have

$$\Delta(Tgx, Tx) \leq \phi\big(D(ggx, gx)\big) = 0.$$

Then $Tgx = Tx$, for some $x \in C(g, T)$. Let $y = gx$, then $y = gy$ and $y = gx \in Tx = Tgx = Ty$. So $y = gy \in Ty$.

Example 2.1 Le t $E = [1, +\infty[$ be endowed with the $\mathcal{F}$ -metric D given by

$$D(x, y) = |x - y|, \quad x, y \in E.$$

With $f(x) = \ln x$ and $a = 0$. Define g and T on E by

$$g : E \to E, \qquad\qquad T : E \to M_F$$

$$x \to g(x) = \frac{x+2}{2}, \qquad\qquad x \to T(x) = \left[1, \frac{3+\sqrt{x}}{2}\right].$$

Then

$$\begin{aligned} \Delta(Tx, Ty) &= \max\left(\sup_{z\in Tx} D(z, Ty), \sup_{w\in Ty} D(w, Tx)\right) \\ &= \left|\frac{\sqrt{x}-\sqrt{y}}{2}\right| \leq \frac{|x-y|}{4} = \frac{1}{2} D(gx, gy), \quad \text{for all } x, y \in E. \end{aligned}$$

Putting $\phi(t) = \frac{t}{2}$, $t \geq 0$, then $\phi \in \Phi$, and we get

$$\Delta(Tx, Ty) \leq \phi\big(D(gx, gy)\big), \quad \text{for all } x, y \in E.$$

Obviously, $Tx \subset g(E)$, $E_T(x) \neq \emptyset, \forall x \in E$ and $g(E) = \left[\frac{3}{2}, +\infty\right[$ is a $\mathcal{F}$-complete subspace of E.

Thus all conditions in Theorem 2.1 are satisfied. Then

1) $g(x) \in Tx$, for all $x \in C(g, T) = \left[1, \frac{3+\sqrt{5}}{2}\right]$.
2) We have $ggx = gx$, for $x = 2 \in C(g, T)$, then g and T have a common fixed point $x = 2 = g(2) \in T2 = \left[1, \frac{3+\sqrt{2}}{2}\right]$.

Theorem 2.2 Let g be a self-map on $\mathcal{F}$ -metric space (E, D) with continuous $f \in \mathcal{F}$and $\alpha \geq 0$ and let $T : E \to M_F$ be a multi-valued mapping such that

$$\Delta\,(T\,x, T\,y) \leq \alpha D\,(gx, gy) + \beta D\,(gx, T\,x) + \delta D\,(gy, T\,y)\,, \qquad \textbf{(2.6)}$$

for all $x, y \in E$, with $\alpha, \beta, \delta \in \mathbb{R}_+$ such that $\alpha + \beta + \delta < 1$, where $Tx \subset g(E)$, for all $x \in E$. If

a) $g(E)$ is a $\mathcal{F}$-complete subspace of E.
b) The real number δ is chosen in order that $f(t) > f(\delta t) + a$ for all $t > 0$, where $f \in \mathcal{F}$and a are given by (D_3). Then
    1) The set $C(g, T)$ is nonempty.
    2) If $ggx = gx$ for some$x \in C(g, T)$, then g and T have a common fixed point.

Proof

1) If $\alpha = \beta = \delta = 0$, it is clear, that there exists $x \in E$, such that $gx \in Tx$. Now if there is at least one non-zero of α, β, δ. Let $x_0 \in E$ be arbitrary and $y_0 = gx_0$. Since $Tx_0 \subset g(E)$, there exists $x_1 \in E$, such that

$$y_1 = gx_1 \in Tx_0.$$

If $\varDelta(Tx_0, Tx_1) = 0$, we have

$$gx_1 \in Tx_0 = Tx_1.$$

Now, if $\varDelta(Tx_0, Tx_1) > 0$, choose $q \in \mathbb{R}$, $1 < q < \frac{1}{\alpha+\beta+\delta}$. By Lemma 1.2, there exists $y_2 \in Tx_1$ such that

$$D(y_1, y_2) \leq q\varDelta(Tx_0, Tx_1).$$

Since $Tx_1 \subset g(E)$, there exists $x_2 \in E$, such that $y_2 = gx_2$. In condition **(2.6)**, we have

$$\begin{aligned} D(gx_1, gx_2) &\leq q\varDelta(Tx_0, Tx_1) \\ &\leq q\big(\alpha D(gx_0, gx_1) + \beta D(gx_0, Tx_0) + \delta D(gx_1, Tx_1)\big) \\ &\leq q\big(\alpha D(gx_0, gx_1) + \beta D(gx_0, gx_1) + \delta D(gx_1, gx_2)\big) \end{aligned}$$

So,

$$D(gx_1, gx_2) \leq \lambda D(gx_0, gx_1), \quad \text{where } 0 < \lambda = \frac{q(\alpha+\beta)}{1-q\delta} < 1.$$

We continue with the same process, if $\varDelta(Tx_1, Tx_2) = 0$, we have

$$gx_2 \in Tx_1 = Tx_2.$$

If $\varDelta(Tx_1, Tx_2) > 0$, we have

$$D(gx_2, gx_3) \leq \lambda D(gx_1, gx_2), \quad \text{where } 0 < \lambda = \frac{q(\alpha+\beta)}{1-q\delta} < 1.$$

We obtain a sequence $(y_n)_n$ in E such that $y_{n+1} = gx_{n+1} \in Tx_n$, and

$$D(y_n, y_{n+1}) \leq \lambda D(y_{n-1}, y_n), \ \forall n \in \mathrm{N}^*, \ 0 < \lambda < 1.$$

By Remark 2.1, it follows that $(gx_n)_n$ is a $\mathcal{F}$ -Cauchy sequence in a complete $\mathcal{F}$ -metric space $(g(E), D)$, hence there exists $x \in E$ such that

$$\lim_{n\to\infty} gx_n = gx.$$

We show that $gx \in Tx$. If $gx \notin Tx$. Since Tx is closed, this implies $D(gx, Tx) > 0$. In condition **(2.6)** and by Remark 1.2, we have

$$f\big(D(gx, Tx)\big) \leq f\big(D(gx, gx_{n+1}) + D(gx_{n+1}, Tx)\big) + a$$
$$\leq f\big(D(gx, gx_{n+1}) + \Delta(Tx_n, Tx)\big) + a.$$
$$\leq f\begin{pmatrix} D(gx, gx_{n+1}) + \alpha D(gx_n, gx) \\ +\beta D(gx_n, Tx_n) + \delta D(gx, Tx) \end{pmatrix} + a$$

Since f is continuous, taking limit as $n \to +\infty$, we have

$$f\big(D(gx, Tx)\big) \leq f\big(\delta D(gx, Tx)\big) + a.$$

Which is a contradiction with respect condition (b). Hence, we obtain $D(gx, Tx) = 0$. Since Tx is closed, then $gx \in Tx$.

2) If $ggx = gx$, for some $x \in C(g, T)$. In condition **(2.6)**, we have

$$\Delta(Tgx, Tx) \leq \alpha D(ggx, gx) + \beta D(ggx, Tgx) + \delta D(gx, Tx) = \beta D(gx, Tgx)$$
$$\leq \beta \Delta(Tx, Tgx).$$

Then

$$\Delta(Tgx, Tx) \leq \beta \Delta(Tx, Tgx) < \Delta(Tx, Tgx).$$

Consequently, $\Delta(Tgx, Tx) < \Delta(Tgx, Tx)$, which is a contradiction.

So, $\Delta(Tgx, Tx) = 0$.

Then, $Tgx = Tx$, for some $x \in C(g, T)$.

Let $y = gx$, then $y = gy$ and $y = gx \in Tx = Tgx = Ty$. So $y = gy \in Ty$.

Example 2.2 Let $E = [1, +\infty[$ be endowed with the $\mathcal{F}$ -metric Dgiven by

$$D(x, y) = |x - y|, \quad x, y \in E.$$

With $f(x) = \ln x$ and $a = 0$. Define g and T on E by

$$g \ : \ E \to E, \qquad\qquad T \ : \ E \to M_F$$
$$x \to g(x) = \frac{x+1}{2}, \qquad x \to T(x) = \left[1, \frac{2+\sqrt{x+3}}{4}\right].$$

For all $x \in E$, we have $\frac{2+\sqrt{x+3}}{4} \leq \frac{x+1}{2}$, then

$$\Delta(Tx, Ty) = \frac{\left|\sqrt{x+3} - \sqrt{y+3}\right|}{4}$$

$$D(gx, Tx) = \inf_{z \in Tx} D(gx, z) = \left|\frac{2x - \sqrt{x+3}}{4}\right|$$

$$D(gy, Ty) = \inf_{z \in Ty} D(gy, z) = \left|\frac{2y - \sqrt{y+3}}{4}\right|.$$

Then

$$\Delta(Tx, Ty) = \left|\frac{\sqrt{x+3} - \sqrt{y+3}}{4}\right| \leq \frac{|x-y|}{16}$$
$$\leq \frac{13}{32} D(gx, gy) + \frac{D(gx, Tx)}{4} + \frac{D(gy, Ty)}{4}.$$

Putting $\alpha = \frac{13}{32}$, $\beta = \delta = \frac{1}{4}$. We get

$$\Delta(Tx, Ty) \leq \frac{13}{32} D(gx, gy) + \frac{1}{4} D(gx, Tx) + \frac{1}{4} D(gy, Ty), \quad \text{for all } x, y \in E.$$

Obviously, $Tx \subset g(E), \forall x \in E$, and $g(E) = [1, +\infty[$ is a $\mathcal{F}$-complete.

Thus all conditions in Theorem 2.2 are satisfied. Then

1) $g(x) \in Tx$, for $x = 1 \in E$.
2) We have $ggx = gx$, for $x = 1 \in C(g, T)$, then g and T have a common fixed point $x = 1 = g(1) \in T1 = [1]$.

Theorem 2.3 Let g be a self-map on $\mathcal{F}$-metric space (E, D) with continuous $f \in \mathcal{F}$ and $\alpha \geq 0$ and let $T : E \to M_F$ be a multi-valued mapping such that

$$\Delta\,(Tx,Ty) \leq \alpha D\,(gx,gy) + LD\,(gy,Tx), \qquad \textbf{(2.7)}$$

for all $x, y \in E$, with $\alpha \in]0, 1[$ and $L \geq 0$, where $Tx \subset g(E)$, for all $x \in E$. If $g(E)$ is a $\mathcal{F}$ -complete subspace of E, then

1) The set $C(g, T)$ is nonempty.
2) If $ggx = gx$ for some $x \in C(g, T)$, then g and T have a common fixed point.

Proof

1) Let $x_0 \in E$ be arbitrary and $y_0 = gx_0$. Since $Tx_0 \subset g(E)$, there exists $x_1 \in E$,such that $y_1 = gx_1 \in Tx_0$. If $\Delta(Tx_0, Tx_1) = 0$, we have $gx_1 \in Tx_0 = Tx_1$. Now, if $\Delta(Tx_0, Tx_1) > 0$, choose $q \in \mathbb{R}$, $1 < q < 1/\alpha$. By Lemma 1.2, there exists $y_2 \in Tx_1$ such that $D(y_1, y_2) \leq q\Delta(Tx_0, Tx_1)$. In condition **(2.7)**, we have
$$D(y_1, y_2) \leq q\Delta(Tx_0, Tx_1) \leq q\big(\alpha D(gx_0, gx_1) + LD(gx_1, Tx_0)\big)$$
$$\leq \lambda D(gx_0, gx_1), \quad 0 < \lambda = \alpha q < 1.$$

Since $Tx_1 \subset g(E)$, there exists $x_2 \in E$, such that $y_2 = gx_2$. Then

$$D(gx_1, gx_2) \leq \lambda D(gx_0, gx_1), \quad 0 < \lambda = \alpha q < 1.$$

We continue with the same process, if $\Delta(Tx_1, Tx_2) = 0$, we have

$$gx_2 \in Tx_1 = Tx_2.$$

If $\Delta(Tx_1, Tx_2) > 0$, we have

$$D(gx_2, gx_3) \leq \lambda D(gx_1, gx_2), \quad 0 < \lambda = \alpha q < 1.$$

We obtain a sequence $(y_n)_n$ in E such that $y_{n+1} = gx_{n+1} \in Tx_n$, and

$$D(y_n, y_{n+1}) \leq \lambda D(y_{n-1}, y_n), \; \forall n \in \mathrm{N}^*, \; 0 < \lambda < 1.$$

By Remark 2.1, it follows that $(gx_n)_n$ is a $\mathcal{F}$ -Cauchy sequence in a complete $\mathcal{F}$ -metric space $(g(E), D)$, hence there exists $x \in E$ such that

$$\lim_{n\to\infty} gx_n = gx.$$

We show that $gx \in Tx$, If $gx \notin Tx$. Since Tx is closed, this implies $D(gx, Tx) > 0$. In condition **(2.7)** and by Remark 1.2, we have

$$\begin{aligned} f\big(D(gx, Tx)\big) &\leq f\big(D(gx, gx_{n+1}) + D(gx_{n+1}, Tx)\big) + a \\ &\leq f\big(D(gx, gx_{n+1}) + \Delta(Tx_n, Tx)\big) + a. \\ &\leq f\big(D(gx, gx_{n+1}) + \alpha D(gx_n, gx) + LD(gx, Tx_n)\big) + a \\ &\leq f\big(D(gx, gx_{n+1}) + \alpha D(gx_n, gx) + LD(gx, gx_{n+1})\big) + a. \end{aligned}$$

Taking limit as $n \to +\infty$, we get $f\big(D(x, Tx)\big) \leq -\infty$, which is a contradiction, hence $D(gx, Tx) = 0$. Since Tx is closed, then $gx \in Tx$.

2) If $ggx = gx$, for some $x \in C(g, T)$. In condition **(2.7)**, we have

$$\Delta(Tgx, Tx) \leq \alpha D(ggx, gx) + LD(gx, Tx) = 0.$$

Then $Tgx = Tx$, for some $x \in C(g, T)$.

Let $y = gx$, then $y = gy$ and

$$y = gx \in Tx = Tgx = Ty.$$

So $y = gy \in Ty$.

Theorem 2.4 Let (E, D) be a complete $\mathcal{F}$ -metric space with continuous $f \in \mathcal{F}$ and $a \geq 0$ and let $T : E \to M_F$ be a multi-valued mapping. If

$$\Delta\,(Tx, Ty) \leq g\big(D\,(hx, hy)\big)D(hx, hy), \tag{2.8}$$

for all $x, y \in E$, with $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a increasing function and $0 \leq g(t) < 1$, for each $t > 0$, where $Tx \subset h(E)$, for all $x \in E$. If $h(E)$ is a $\mathcal{F}$ -complete subspace of E, then

1) The set $C(h, T)$ is nonempty.
2) If $hhx = hx$ for some $x \in C(h, T)$, then h and T have a common fixed point.

Proof

1) Let $x_0 \in E$ be arbitrary and $y_0 = gx_0$. Since $Tx_0 \subset h(E)$,there exists $x_1 \in E$,such that $y_1 = hx_1 \in Tx_0$. If $\Delta(Tx_0, Tx_1) = 0$, we have $hx_1 \in Tx_0 = Tx_1$. Now, if $\Delta(Tx_0, Tx_1) > 0$, then $g\big(D(hx_0, hx_1)\big) > 0$. Choose $q \in \mathrm{R}$, $1 < q < \frac{1}{g(D(hx_0, hx_1))}$. By Lemma 1.2, there exists $y_2 \in Tx_1$ such that $D(y_1, y_2) \leq q\Delta(Tx_0, Tx_1)$. In condition **(2.8)**, we have

$$\begin{aligned} D(y_1, y_2) &\leq q\Delta(Tx_0, Tx_1) \leq q\left(g\big(D(hx_0, hx_1)\big)\, D(hx_0, hx_1)\right) \\ &\leq \lambda D(hx_0, hx_1), \quad 0 < \lambda = qg\big(D(hx_0, hx_1)\big) < 1 \end{aligned}$$

Since $Tx_1 \subset h(E)$, there exists $x_2 \in E$, such that $y_2 = hx_2$. Then

$$D(hx_1, hx_2) \leq \lambda D(hx_0, hx_1), \quad 0 < \lambda < 1.$$

Again, if $\Delta(Tx_1, Tx_2) = 0$, we have $y_2 = hx_2 \in Tx_1 = Tx_2$. If

$$\Delta(Tx_1, Tx_2) > 0,$$

then $g\big(D(hx_1, hx_2)\big) > 0$, By Lemma 1.2, there exists $y_3 \in Tx_2$ such that $D(y_2, y_3) \leq q\Delta(Tx_1, Tx_2)$. Since$g$ is a increasing function and

$$D(hx_1, hx_2) \leq \lambda D(hx_0, hx_1) < D(hx_0, hx_1).$$

Then

$$D(y_2, y_3) \leq q\Delta(Tx_1, Tx_2) \leq q\left(g\big(D(hx_1, hx_2)\big) D(hx_1, hx_2)\right)$$
$$\leq q\left(g\big(D(hx_0, hx_1)\big) D(hx_1, hx_2)\right) = \lambda D(hx_1, hx_2).$$

Since $Tx_2 \subset h(E)$, there exists $x_3 \in E$, such that $y_3 = hx_3$. Then

$$D(hx_2, hx_3) \leq \lambda D(hx_1, hx_2), \quad 0 < \lambda < 1.$$

We obtain a sequence $(y_n)_n$ in E such that $y_{n+1} = hx_{n+1} \in Tx_n$, $n \in \mathbb{N}$ and

$$D(y_n, y_{n+1}) \leq \lambda D(y_{n-1}, y_n), \ \forall n \in \mathbb{N}^*, \ 0 < \lambda < 1.$$

By Remark 2.1, it follows that $(y_n)_n$ is a $\mathcal{F}$ -Cauchy sequence in a complete $\mathcal{F}$-metric space $(h(E), D)$, hence there exists $x \in E$ such that $\lim_{n\to\infty} hx_n = hx$. We show that $hx \in Tx$, If $hx \notin Tx$. Since Tx is closed, this implies $D(hx, Tx) > 0$. In condition **(2.8)** and by Remark 1.2, we have

$$f\big(D(hx, Tx)\big) \leq f\big(D(hx, hx_{n+1}) + D(hx_{n+1}, Tx)\big) + a$$
$$\leq f\big(D(hx, hx_{n+1}) + \Delta(Tx_n, Tx)\big) + a.$$
$$\leq f\left(D(hx, hx_{n+1}) + g\big(D(hx_n, hx)\big) D(hx_n, hx)\right) + a$$
$$\leq f\big(D(hx, hx_{n+1}) + D(hx_n, hx)\big) + a.$$

Taking limit as $n \to +\infty$, we get $f\big(D(hx, Tx)\big) \leq -\infty$, which is a contradiction, hence $D(hx, Tx) = 0$. Since Tx is closed, then $hx \in Tx$.

2) If $hhx = hx$, for some $x \in C(h, T)$. In condition **(2.8)**, we have

$$\Delta(Thx, Tx) \leq g\big(D(hhx, hx)\big) D(hhx, hx) = 0.$$

Then $Thx = Tx$, for some $x \in C(h, T)$. Let $y = hx$, then $y = hy$ and $y = hx \in Tx = Thx = Ty$. So, $y = hy \in Ty$.

Example 2.3 Let $E = \mathbb{R}_+$ be endowed with the $\mathcal{F}$ -metric D given by

$$D(x, y) = |x - y|, \quad x, y \in E.$$

With $f(x) = \ln x$ and $a = 0$. Define g and T on E by

$$h : E \to E, \qquad\qquad T : E \to M_F$$
$$x \to h(x) = \frac{x+3}{2}, \qquad\qquad x \to T(x) = \left[0, \frac{4+\sqrt{x+1}}{2}\right].$$

Let g be a mapping on $\mathbb{R}_+$ defined by

$$g \ : \mathbb{R}_+ \to \mathbb{R}_+$$
$$t \to g(t) = \frac{t+1}{t+2}$$

Then g is a increasing function and $0 \leq g(t) < 1.$ We obtain

$$\Delta(Tx,\, Ty) = \frac{\left|\sqrt{x+1} - \sqrt{y+1}\right|}{2}, \quad D(hx,\, hy) = \frac{|x-y|}{2}$$
$$g\big(D(hx,\, hy)\big) = \frac{|x-y|+2}{|x-y|+4} \geq \frac{1}{2}$$

Then

$$\Delta(Tx,\, Ty) = \left|\frac{\sqrt{x+1} - \sqrt{y+1}}{2}\right| \leq \frac{|x-y|}{4} = \frac{1}{2} D(hx,\, hy)$$
$$\leq g\big(D(hx,\, hy)\big)\, D(hx,\, hy).$$

We get

$$\Delta(Tx,\, Ty) \leq g\big(D(hx,\, hy)\big)\, D(hx,\, hy), \quad \text{for all } x,\, y \in E.$$

Obviously, $Tx \subset h(E), \forall x \in E$,and $h(E) = \left[\frac{3}{2}, +\infty\right[$ is a $\mathcal{F}$ -complete subspace of E .

Thus all conditions in Theorem 2.4 are satisfied. Then

1) $h(x) \in Tx,$ for all $x \in C(h,\, T) = [0,\, 3].$
2) We have $hhx = hx,$ for $x = 3 \in C(h,\, T),$ then $h$and T have a common fixed point $x = 3 = h(3) \in T3 = [0,\, 3].$

We present the following consequences of Theorems 2.1, 2.2, 2.3, 2.4 respectively.

Theorem 2.5 Let (E, D) be a complete $\mathcal{F}$ -metric space with continuous $f \in \mathcal{F}$ and $a \geq 0.$ Let $(M_F,\, \Delta)$ be $\mathcal{F}$ -metric space. Suppose $T \ : \ E \to M_F$ is a multi-valued mapping such that

$$\Delta\,(Tx, Ty) \leq \ \phi\,\big(D\,(x, y)\big), \qquad \textbf{(2.9)}$$

for all $x,\, y \in E,$ where $\phi \in \Phi.$ Suppose that the following assertion hold:

For each $x \in E,$ the set

$$E_T(x) = \{y \in Tx; \ \ L(Tx,\, x) \leq qD(y,\, x) \ \text{ for some } q > 1\}$$

is nonempty. Then, there exists an element x in $E,$ such that $x \in T(x).$

Proof Putting $g = I_E$ in Theorem 2.1, we get the result.

Theorem 2.6 ([6, Proposition 4]) Let $(E,\, D)$ be a complete $\mathcal{F}$ -metric space with continuous $f \in \mathcal{F}$ and $a \geq 0.$ Furthermore, let M_F be the set of all nonempty $\mathcal{F}$ -closed and bounded subsets of E and let Δ be the $\mathcal{F}$ -Hausdorff distance which turns $(M_F,\, \Delta)$ into an $\mathcal{F}$ -metric space. Suppose $T \ : \ E \to M_F$ and $0 < k < 1$ are such that

$$\Delta\,(Tx, Ty) \leq \ kD\,(x, y), \qquad \textbf{(2.10)}$$

for every $x,\, y \in E.$ Then, there exists an element $x \in E,$ such that $x \in T(x).$

Proof Putting $g = I_E$, and $\alpha \in]0, 1[$, $L = 0$ in Theorem 2.3, we get the result.

Theorem 2.7 ([7]) Let (E, D) be a complete $\mathcal{F}$ -metric space with continuous $f \in \mathcal{F}$ and $a \geq 0$ and let $T : E \to M_F$ be a multi-valued mapping. If

$$\Delta (Tx, Ty) \leq \alpha D(x, y) + \beta D(x, Tx) + \delta D(y, Ty), \quad \textbf{(2.11)}$$

for all $x, y \in E$, with $\alpha, \beta, \delta \in \mathbb{R}_+$ such that $\alpha + \beta + \delta < 1$. Then, there exists an element x in E,such that $x \in T(x)$ if the following condition is satisfied: The real number δ is chosen in order that $f(t) > f(\delta t) + a$ for all $t > 0$, where $f \in \mathcal{F}$ and a are given by (D_3).

Proof Putting $g = I_E$ in Theorem 2.2, we get the result.

Theorem 2.8 ([1, Theorem 3 in $\mathcal{F}$ -metric space]) Let (E, D) be a complete $\mathcal{F}$ -metric space with continuous $f \in \mathcal{F}$ and $a \geq 0$ and let $T : E \to M_F$ be a multi-valued mapping. If

$$\Delta (Tx, Ty) \leq \alpha D(x, y) + LD(y, Tx), \quad \textbf{(2.12)}$$

for all $x, y \in E$, with $\alpha \in]0, 1[$ and $L \geq 0$. Then, there exists an element x in E, such that $x \in T(x)$.

Proof Putting $g = I_E$ in Theorem 2.3, we get the result.

Theorem 2.9 ([4, in $\mathcal{F}$ -metric space]) Let (E, D) be a complete $\mathcal{F}$ -metric space with continuous $f \in \mathcal{F}$ and $a \geq 0$ and let $T : E \to M_F$ be a multi-valued mapping. If

$$\Delta(Tx, Ty) \leq g\big(D(x, y)\big) D(x, y), \quad \textbf{(2.13)}$$

for all $x, y \in E$, with $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a increasing function and $0 \leq g(t) < 1$, for each $t > 0$. Then, there exists an element x in E such that $x \in T(x)$.

Proof Putting $h = I_E$ in Theorem 2.4, we get the result.

3-Application

Definition 3.1 We say that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a sub additive function if

$$\int_0^{\varepsilon+\mu} \psi(t)dt \leq \int_0^{\varepsilon} \psi(t)dt + \int_0^{\mu} \psi(t)dt$$

for all $\varepsilon > 0$ and all $\mu > 0$.

Let Y be the set of functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

1- ψ is a Lebesgue integrable which is non negative and satisfies $\int_0^{\varepsilon} \psi(t)dt > 0$ for each $\varepsilon > 0$.

2- ψ is a sub additive.

3- If $f \in \mathcal{F}$ a continuous function, there exists a continuous function $f_1 \in \mathcal{F}$ such that

$$f(\varepsilon) = f_1\left(\int_0^{\varepsilon} \psi(t)dt\right), \forall \varepsilon > 0.$$

Remark 3.1 The set $Y \neq \emptyset$. There exists $\psi \in Y$such that $\psi(t) = \frac{1}{1+t}$, $t \geq 0$.

If $t = 0$, it's clear, if $t > 0$, then $\int_0^\varepsilon \psi(t)dt = \ln(1+\varepsilon) > 0$, and

$$\int_0^{\varepsilon+\mu} \psi(t)dt = \int_0^{\varepsilon+\mu} \frac{1}{1+t}dt = \ln(1+\varepsilon+\mu)$$
$$\leq \ln(1+\varepsilon)\,(1+\mu) = \ln(1+\varepsilon) + \ln(1+\mu)$$
$$\leq \int_0^\varepsilon \psi(t)dt + \int_0^\mu \psi(t)dt\,.$$

Let $f \in \mathcal{F}$ be, we define $f_1 \;:\;]0,\infty[\, \to \mathbb{R}$, by

$$f_1(x) = f\big(-1 + \exp(x)\big).$$

We have

$$f_1\left(\int_0^\varepsilon \psi(t)dt\right) = f_1\big(\ln(1+\varepsilon)\big) = f\left(-1 + \exp\big(\ln(1+\varepsilon)\big)\right) = f(\varepsilon),$$

it's clear that f_1 is non-decreasing, and if f is continuous, then f_1 is continuous. Now, for every sequence $(s_n) \subset\,]0,\infty[$, we have

$$\lim_{n\to+\infty} s_n = 0 \text{ if and only if } \lim_{n\to+\infty}\big(-1+\exp(s_n)\big) = 0$$
$$\text{if and only if } \lim_{n\to+\infty} f_1(s_n) = \lim_{n\to+\infty} f\big(-1+\exp(s_n)\big) = -\infty.$$

Lemma 3.1 Let $(E,\, D)$ be an $\mathcal{F}$ -metric space with $(f,\, a) \in \mathcal{F} \times [0,\infty[$, and let $\widehat{D} \;:\; E \times E \to [0,\infty[$ be a mapping given by

$$\widehat{D}(x,\, y) = \int_0^{D(x,y)} \psi(t)dt,$$

for all $x,\, y \in E$, where $\psi \in Y$. There exists a function $f_1 \in \mathcal{F}$ such that $\big(E,\, \widehat{D}\big)$ is a $\mathcal{F}$ -metric space with $(f_1,\, a) \in \mathcal{F} \times [0,\infty[$

Proof Let $\psi \in Y$, there exists a continuous function $f_1 \in \mathcal{F}$ such that

$$f(\varepsilon) = f_1\left(\int_0^\varepsilon \psi(t)dt\right), \forall \varepsilon > 0.$$

For all $(x,\, y) \in E^2$, we have

1) $\widehat{D}(x,\, y) = 0$ if and only if $D(x,\, y) = 0$ if and only if $x = y$.
2) $\widehat{D}(x,\, y) = \widehat{D}(y,\, x)$.
3) For every $N \in \mathbb{N}$, $N \geq 2$ and for all $(v_i)_{i=1}^N \subset E$ with $(v_1,\, v_N) = (x,\, y)$, we obtain

$$\widehat{D}(x,\, y) > 0,\ \ then\ \ D(x,\, y) > 0$$

$$\text{so, } f_1\left(\widehat{D}(x,\, y)\right) = f_1\left(\int_0^{D(x,y)} \psi(t)dt\right) = f\big(D(x,\, y)\big)$$

$$\leq f\left(\sum_{i=1}^{N-1} D(v_i,\, v_{i+1})\right) + a$$

$$= f_1\left(\int_0^{\sum_{i=1}^{N-1} D(v_i, v_{i+1})} \psi(t)dt\right) + a$$

$$\leq f_1\left(\sum_{i=1}^{N-1} \int_0^{D(v_i, v_{i+1})} \psi(t)dt\right) + a$$

$$= f_1\left(\sum_{i=1}^{N-1} \widehat{D}(v_i,\, v_{i+1})\right) + a.$$

Then $\widehat{D}$ is an $\mathcal{F}$-metric on E with $(f_1,\, a) \in \mathcal{F} \times [0, \infty[$

Lemma 3.2 Let $(E,\, D)$ be an $\mathcal{F}$ -metric space with continuous function $f \in \mathcal{F}$ and $a \geq 0$, and let $\hat{\Delta}:\ M_F \times M_F \to [0, \infty[$ be a mapping is defined by

$$\hat{\Delta}(A,\, B) = \int_0^{\Delta(A,B)} \psi(t)dt,$$

for all $A,\, B \in M_F$, where $\psi \in Y$, and Δ is a $\mathcal{F}$ -metric space with $(f,\, a) \in \mathcal{F} \times [0, \infty[$, given by **(1.1)**. There exists a continuous function $f_1 \in \mathcal{F}$ such that $\left(M_F,\, \hat{\Delta}\right)$ is a $\mathcal{F}$ -metric space with $(f_1,\, a) \in \mathcal{F} \times [0, \infty[$ and

$$\hat{\Delta}(A,\, B) = \max\left(\hat{L}(A,\, B),\, \hat{L}(B,\, A)\right),\ \ \forall A,\, B \in M_F,$$

where

$$\hat{L}(A,\, B) = \sup_{x \in A} \widehat{D}(x,\, B)$$

Proof By Lemma 3.1, $\left(M_F,\, \hat{\Delta}\right)$ is a $\mathcal{F}$ -metric space with continuous function $f_1 \in \mathcal{F}$ and $a \geq 0$, and $\widehat{D}$ is defined by

$$\widehat{D}(x,\, y) = \int_0^{D(x,y)} \psi(t)dt,$$

then

$$\widehat{D}(x,\, B) = \inf_{y \in B} \widehat{D}(x,\, y) = \inf_{y \in B} \int_0^{D(x,y)} \psi(t)dt$$

$$= \int_0^{\inf_{y \in B} D(x,y)} \psi(t)dt = \int_0^{D(x,B)} \psi(t)dt\,.$$

Thus, we have

$$\hat{\Delta}(A,B)=\int_0^{\Delta(A,B)}\psi(t)dt=\int_0^{\max(L(A,B),L(B,A))}\psi(t)dt$$
$$=\max\left(\int_0^{L(A,B)}\psi(t)dt\,,\int_0^{L(B,A)}\psi(t)dt\right)$$
$$=\max\left(\int_0^{\sup_{x\in A}D(x,B)}\psi(t)dt\,,\int_0^{\sup_{x\in B}D(x,A)}\psi(t)dt\right)$$
$$=\max\left(\sup_{x\in A}\int_0^{D(x,B)}\psi(t)dt\,,\sup_{x\in B}\int_0^{D(x,A)}\psi(t)dt\right)$$
$$=\max\left(\sup_{x\in A}\widehat{D}(x,B),\sup_{y\in B}\widehat{D}(x,A)\right)$$
$$=\max\left(\hat{L}(A,B),\hat{L}(B,A)\right)$$

Theorem 3.1 Let g be a self-map on $\mathcal{F}$ -metric space (E,D) with continuous function $f\in\mathcal{F}$ and $a\geq 0$ and let $T\ :\ E\to M_F$ be a multi-valued mapping such that.

$$\int_0^{\Delta(Tx,Ty)}\psi(t)dt\leq\phi\left(\int_0^{D(gx,gy)}\psi(t)dt\right),\qquad\textbf{(3.1)}$$

For all $x,y\in E$, with $\phi\in\Phi$, where $\psi\in Y$ and $Tx\subset g(E)$, for all $x\in E$. Suppose that the following assertions hold:

a- For each $x\in E$, the set

$$E_T(x)=\{y\in Tx;\ \ L(Tx,gx)\leq qD(y,gx)\ \text{ for some } q>1\}$$

is nonempty.

b- If $g(E)$ is a $\mathcal{F}$ -complete subspace of E, then

1) The set $C(g\cap T)$ is nonempty.
2) If $ggx=gx$ for all $x\in C(g\cap T)$, then g and T have a common fixed point.

Proof The inequality **(3.1)** becomes

$$\hat{\Delta}(Tx,Ty)\leq\phi\left(\widehat{D}(gx,gy)\right)$$

By Lemmas 3.1 and 3.2, $\widehat{D}$ is an $\mathcal{F}$ -metric on E, and $\hat{\Delta}$ is an $\mathcal{F}$ -metric on M_F. Now the proof follows directly from theorem 2.1.

Theorem 3.2 Let g be a self-map on $\mathcal{F}$ -metric space (E,D) with continuous function $f\in\mathcal{F}$ and $a\geq 0$ and let $T\ :\ E\to M_F$ be a multi-valued mapping such that

$$\int_0^{\Delta(Tx,Ty)}\psi(t)dt\leq\alpha\int_0^{D(gx,gy)}\psi(t)dt$$
$$+\beta\int_0^{D(gx,Tx)}\psi(t)dt+\delta\int_0^{D(gy,Ty)}\psi(t)dt,\qquad\textbf{(3.2)}$$

for all $x,y\in E$, with $\alpha,\beta,\delta\in\mathbb{R}_+$ such that $\alpha+\beta+\delta<1$, where $\psi\in Y$ and $Tx\subset g(E)$, for all $x\in E$. If

a) $g(E)$ is a $\mathcal{F}$ -complete subspace of E.

b) The real number δ is chosen in order that $f(t) > f(\delta t) + a$ for all where $f \in \mathcal{F}$ and a are given by (D_3). Then

1) The set $C(g, T)$ is nonempty.
2) If $ggx = gx$ for some $x \in C(g, T)$, then g and T have a common fixed point.

Proof The inequality **(3.2)** becomes

$$\hat{\Delta}(Tx, Ty) \leq \alpha\widehat{D}(gx, gy) + \beta\widehat{D}(gx, Tx) + \delta\widehat{D}(gy, Ty).$$

By Lemmas 3.1 and 3.2, $\widehat{D}$ is an $\mathcal{F}$ -metric on E, and $\hat{\Delta}$ is an $\mathcal{F}$ –metric on M_F Now the proof follows directly from theorem 2.2.

Theorem 3.3 Let g be a self-map on $\mathcal{F}$ -metric space (E, D) with continuous function $f \in \mathcal{F}$ and $a \geq 0$ and let $T : E \to M_F$ be a multi-valued mapping such that

$$\int_0^{\Delta(Tx,Ty)} \psi(t)dt \leq \alpha \int_0^{D(gx,gy)} \psi(t)dt + L \int_0^{D(gx,Tx)} \psi(t)dt \qquad \textbf{(3.3)}$$

For all $x, y \in E$, with $\alpha \in]0, 1[$ and $L \geq 0$, where $\psi \in Y$ and $Tx \subset g(E)$, for all $x \in E$. If $g(E)$ is a $\mathcal{F}$-complete subspace of E, then

- The set $C(g, T)$ is nonempty.
- If $ggx = gx$ for some $x \in C(g, T)$, then g and T have a common fixed point.

Proof The inequality **(3.3)** becomes

$$\hat{\Delta}(Tx, Ty) \leq \alpha\widehat{D}(gx, gy) + L\widehat{D}(gy, Tx).$$

By Lemmas 3.1 and 3.2, $\widehat{D}$ is an $\mathcal{F}$ -metric on E, and $\hat{\Delta}$ is an $\mathcal{F}$ -metric on M_F. Now the proof follows directly from theorem 2.3.

Theorem 3.4 Let (E, D) be a complete $\mathcal{F}$ -metric space with continuous function $f \in \mathcal{F}$ and $a \geq 0$ and let $T : E \to M_F$ be a multi-valued mapping. If

$$\int_0^{\Delta(Tx,Ty)} \psi(t)dt \leq g\left(\int_0^{D(hx,hy)} \psi(t)dt\right). \int_0^{D(hx,hy)} \psi(t)dt, \qquad \textbf{(3.4)}$$

for all $x, y \in E$, with $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a increasing function and $0 \leq g(t) < 1$, for each $t > 0$, where $\psi \in Y$ and $Tx \subset h(E)$, for all $x \in E$. If $h(E)$ is a $\mathcal{F}$ -complete subspace of E, then

1) The set $C(h, T)$ is nonempty.
2) If $hhx = hx$ for some $x \in C(h, T)$, then h and T have a common fixed point.

Proof The inequality **(3.4)** becomes

$$\hat{\Delta}(Tx, Ty) \leq g\left(\widehat{D}(hx, hy)\right) \widehat{D}(hx, hy).$$

By Lemmas 3.1 and 3.2, $\widehat{D}$ is an $\mathcal{F}$ -metric on E, and $\hat{\Delta}$ is an $\mathcal{F}$ -metric on M_F. Now the proof follows directly from theorem 2.4.

References

1. M.Berinde , V.Berinde (2007). On a general class of multi-valued weakly Picard mappings. J. Math. Anal. Appl. 326 - 772--782. https://doi.org/10.1016/j.jmaa.2006.03.016
2. Jahangir, F., Haghmaram, P. & Nourouzi, K. (2021). A note on F-metric spaces. *J. Fixed Point Theory Appl.* **23**, 2. https://doi.org/10.1007/s11784-020-00836-y
3. M.Jleli, B.Samet. (2018). On new generalization of metric Spaces. Journal of Fixed Point Theory and Applications, 20, 128. https://doi.org/10.1007/s11784-018-0606-6 .
4. H.Kaneko. (1988). Generalized contractive multi-valued mappings and their fixed points. Math. Japon. 33, 57-64. https://cir.nii.ac.jp/articles

5. T.Lazar, G.Mot, G.Petrusel , and S.Szentesi (Volume 2010). The Theory of Reich's Fixed Point Theorem for Multivalued Operators. Hindawi Publishing Corporation Fixed Point Theory and Applications, Article ID 178421, 10 pages. https://doi:10.1155/2010/178421.
6. Z.D. Mitrovic, H. Aydi, N. Hussain, A. Mukheimer. Reich, Junck, and Berinde. (28 April 2019). Common fixed point on F -Metric Spaces and an application. Mathematics, 7(387), 1-10 http://www.mdpi.com/journal/mathematics .
7. S. Reich. (1972). Fixed points of contractive functions, Bollettino della Unione Matematica Italiana, vol. 5, pp. 26--42. https://api.semanticscholar.org/CorpusID:115383852
8. B.Samet, M.Turinici. (2012). Fixed Point Theorems on a metric space endowed with an arbitrary binary relation and applications. Communications in Mathematical Analysis, Volume 13, Number 2, pp. 82—97. ISSN 1938-9787.

W. Sintunavarat, P. Kumam. (2011). Weak condition for generalized multi-valued (f,α,β)-weak contraction mappings . Applied Mathematics Letters 24 - 460-465. https://doi.org/10.1016/j.aml.2010.10.042

9. W.Sintunavarat, P.Kumam (2012). Common fixed point theorem for cyclic generalized multi-valued contraction mappings Applied Mathematics Letters 25. 1849-1855. https://doi.org/10.1016/j.aml.2012.02.045