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# Variational Analysis of a Dynamic Contact Problem with Wear and Damage Involving Viscoelastic Materials with Long-Term Memory

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**Abstract**
We investigate a mathematical problem involving dynamic interaction between a viscoelastic body with long-term memory loss and an obstruction. The contact is frictional and bilateral, with a moving rigid base, resulting in wear of the contacting surface. The problem is expressed as a coupled system, with a hyperbolic quasi-variational inequality for displacement and a parabolic variational inequality for damage. We define a variational formulation for the model and demonstrate the existence of a single weak solution to the problem. The material behaviour is explained using a viscoelastic constitutive law that includes long-term memory and damage. Elastic deformations induce material deterioration, which is depicted by a parabolic inclusion. The proof is based on classical existence.

**Keywords:** Viscoelastic. Long-term memory. Wear. Damage. Fixed point. Hyperbolic. Quasi-variational inequality.

## 1. Introduction

Problems of contact with or without friction, involving deformable or non-deformable bodies, occur in many ways, both in industrial fields and in everyday life. In view of the importance and the multitude of these phenomena, extensive studies have been undertaken, so the literature concerning contact mechanics is vast and covers as many diserent subjects as are modeling, mathematical analysis. Often, these are models in the form of variational equations or inequalities with non-standard limit terms. The aim of these studies is to place these results and basic computational methods in a unified format that can be accessed by specialists and graduate students. Steps in this direction have been accomplished in many monographs; see, for instance, Han and Sofonea (2000), Li and Liu (2010) and the references therein. Our research is supported by studies in this field; more informations are avaialable in this references: Hamidat and Aissaoui (2021), Hamidat and Aissaoui (2022), Hamidat and Aissaoui (2023).

As a generalization of the communication problem discussed in Chau, Shillor, and Sofonea (2004), Chau *et al*. Chau, Fernández, Han, and Sofonea (2003) studied a dynamic frictionless contact problem and gave a fully discrete scheme for solving it. Bartosz (2006) considered a dynamical viscoelastic contact problem to modify the model treated by Ciulcu *et al*. (2002). In particular, Bartosz (2006) proved the existence of weak solutions to the dynamical viscoelastic contact problem with wear by using the surjectivity result for a class of pseudomonotone operators in the framework of hemivariational inequalities.

Recently, Cocou (2015) extended the static contact problem considered by Rabier *et al*. (2000) to a dynamic viscoelastic contact problem with friction and obtained the existence and uniqueness of the weak solution for such a problem. And as far as we know, there is a study on the dynamic viscoelastic contact problem with friction and wear in Chen, Huang, and Xiao (2020). The aim of this paper is to make a new attempt in this direction. And that, by introducing material damage. The subject of damage is extremely important in design engineering because it directly affcts the useful life of the structure or component being designed. There is a very large engineering literature on it. Models taking into account the influence of internal damage of matter on the contact process have been studied mathematically. New general models for damage were learned in Chau *et al*. (2003), and Chau *et al*. (2004) from the principle of virtual power. The mathematical analysis of one-dimensional problems can be found in Chau *et al*. (2004). The damage function $\beta$ varies between 0 and 1. When $\beta = 1$ there is no damage in the material, when $\beta = 0$ the material is completely damaged, when $0 \prec \beta \prec 1$ the damage is partial. In this paper the relation used to model the evolution of the damage field is as follows $\dot{\beta} - k\Delta\beta + \partial\varphi_K(\beta) \ni \psi(\varepsilon(u), \beta)$ where $K$ is the set of admissible damage test functions, $\varphi$ being the source function of the damage. In this work, we consider a version of a dynamic model describing a contact problem frictional with wear and damage in viscoelastic with long term memory body.

The paper is structured as follows. In Section 2, we introduce some essential preliminaries. In Section 3, we present the mechanical problem, list the assumptions on the data, and give the variational formulation of the problem. In Section 4, we present the proof of the Theorem 4. The arguments for the proof are based on the hyperbolic quasi-variational inequality, parabolic inequalities and Banach is fixed point theorem.

## 2. Preliminaries

In this section, we present some essential tools for our main results. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\Omega \subset \mathbb{R}^d (d = 2,3)$ with a smooth boundary $\partial\Omega = \Gamma$. The boundary $\partial\Omega$ is divided into three disjoint measurable parts $\Gamma_1, \Gamma_2$ and $\Gamma_3$.

We denote by $\boldsymbol{\nu} = (\nu_i)$ the unit outward normal vector and by $x \in \overline{\Omega} = \Omega \cup \partial\Omega$ the position vector. Note that the indications $i, j$ run from 1 to $d$, unless stated otherwise, the summation convention over repeated indications is used. For simplicity, we do not indicate explicitly the dependence of the variables on $x$. The inner products and norms for $\mathbb{R}^d$ and $\mathbb{S}^d$ are denoted by

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i , \quad \| \mathbf{v} \|_{\mathbb{R}^d} = (\mathbf{v}, \mathbf{v})^{\frac{1}{2}} \ \text{ for all } \ \mathbf{u} = (u_i), \mathbf{v} = (v_i) \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma}.\boldsymbol{\tau} = \sigma_{ij}\tau_{ij} , \quad \| \boldsymbol{\tau} \|_{\mathbb{S}^d} = (\boldsymbol{\tau}, \boldsymbol{\tau})^{\frac{1}{2}} \ \text{ for all } \ \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

respectively.

The normal and tangential components of the displacement $\mathbf{u}$ on $\Gamma$ is denoted by $u_\nu = u.\nu$ and $\mathbf{u}_\tau = \mathbf{u} - u_\nu \boldsymbol{\nu}$. The similar notation is used for $\dot{u}_\nu$ and $\dot{\mathbf{u}}_\tau$ which are the normal and tangential velocities on the boundary. The normal and tangential components of stress field $\boldsymbol{\sigma}$ on the boundary are defined by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}).\boldsymbol{\nu}$ and $\boldsymbol{\sigma_\tau} = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$, respectively.

We also use the following notations

$$H = L^2(\Omega)^d = \{\mathbf{u} = (u_i) | u_i \in L^2(\Omega)\}, \quad H_1 = \{\mathbf{u} = (u_i) | \varepsilon(\mathbf{u}) \in \mathcal{H}\},$$

$$\mathcal{H} = \{\boldsymbol{\sigma} = (\sigma_{ij}) | \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}, \quad \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} | Div\boldsymbol{\sigma} \in H\}.$$

The operators of deformation $\varepsilon$ and divergence $Div$ are defined as follows

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad Div\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The spaces $H$, $H_1$, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces with the canonical inner products defined as follows

$$(\mathbf{u},\mathbf{v})_H = \int u_i v_i dx \quad \forall \mathbf{u},\mathbf{v} \in H,$$

$$(\mathbf{u},\mathbf{v})_{H_1} = (\mathbf{u},\mathbf{v})_H + (\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathcal{H}}, \forall \mathbf{u},\mathbf{v} \in H_1,$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int \sigma_{ij}\tau_{ij} dx \quad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathcal{H},$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (Div\boldsymbol{\sigma}, Div\boldsymbol{\tau})_H, \quad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathcal{H}_1.$$

The associated norm in the space $H$, $H_1$, $\mathcal{H}$ and $\mathcal{H}_1$, is denoted by $\|.\|_H$, $\|.\|_{H_1}$, $\|.\|_{\mathcal{H}}$ and $\|.\|_{\mathcal{H}_1}$, respectively.

When $\boldsymbol{\sigma}$ is a regular function, the following Green is type formula holds,

$$\left(\boldsymbol{\sigma},\varepsilon(\mathbf{v})\right)_{\mathcal{H}} + (Div\boldsymbol{\sigma},\mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma\nu}.\mathbf{v}\, da, \quad \forall \mathbf{v} \in H_1. \tag{1}$$

For the displacement field we need the closed subspace of $H_1$ defined by

$$V = \{\mathbf{u} \in H_1 | \mathbf{u} = \mathbf{0} \text{ on } \Gamma_1\},$$

and we denote by $V'$ the dual space of $V$. We can consider the duality pairing $(.,.)_{V'\times V}$ as continuous extension of the inner product $(.,.)_H$ on $H$, i.e.,

$$(\mathbf{u},\mathbf{v})_H = (\mathbf{u},\mathbf{v})_{V'\times V}, \quad \forall \mathbf{u} \in H, \forall \mathbf{v} \in V.$$

The notation $\mathcal{L}(V,V')$ stands for the space of linear continuous operators from a Banach space $V$ to a Banach space $V'$.

Since meas $(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_k > 0$, that depends only on $\Omega$ and $\Gamma_1$, such that

$$\| \varepsilon(\mathbf{v}) \|_{\mathcal{H}} \geq C_k \| \mathbf{v} \|_{H^1(\Omega)^d}, \quad \forall \mathbf{v} \in V,$$

We define inner product on $V$ by

$$(\mathbf{u},\mathbf{v})_V = (\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \| \mathbf{v} \|_V = \| \varepsilon(\mathbf{v}) \|_{\mathcal{H}}, \quad \forall \mathbf{u},\mathbf{v} \in V, \tag{2}$$

and let $\|.\|_V$ be the associated norm. It follows that $\|.\|_{H^1(\Omega)^d}$ and $\|.\|_V$ are equivalent norms on $V$ and therefore $(V,(.,.)_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $\tilde{C}_0$, depending only on $\Omega$, $\Gamma_1$ and $\Gamma_3$, such that

$$\| \mathbf{v} \|_{L^2(\Gamma_3)^d} \leq \tilde{C}_0 \| \mathbf{v} \|_V, \quad \forall \mathbf{v} \in V. \tag{3}$$

We recall some spaces $W^{k,p}(0,T;V)$, $H^k(0,T;V)$ and $C(0;T;V)$ for a Banach space $V$ equipped with the norm $\|.\|_V$ for $1 < p < +1$ and $k \geq 1$. Let $W^{k,p}(0,T;V)$ be the space of all functions from $[0,T]$ to $V$ with the norm

$$\| \delta \|_{W^{k,p}(0,T;V)} = \begin{cases} \left(\int_0^T \sum_{1\leq l\leq k} \left\|\partial_t^l \delta\right\|_V^p dt\right)^{1/p}, & \text{if } 1 \leq p < +\infty, \\ \max_{0\leq l\leq k_0 \leq t\leq T} \sup_t \left\|\partial_t^l \delta\right\|_V, & \text{if } p = +\infty. \end{cases}$$

When $p = 2$ or $k = 0$, $W^{k,2}([0,T];V)$ is written as $H^k([0,T];V)$ or $L^p([0,T];V)$, respectively. Let $C([0,T];V)$ denote the space of all continuous functions from $[0,T]$ to $V$ with the norm

$$\| u \|_{C([0,T];V)} = \max_{t\in[0,T]} \| u(t) \|_V.$$

Clearly, $C([0,T];V)$, $W^{k,p}([0,T];V)$ and $H^k([0,T];V)$ are all Banach spaces when $V$ is a Banach space.

The following existence, uniqueness and regularity result is carried out in the next Theorem and is based on the abstract result for hyperbolic quasi-variational inequality. Let $V$ be a Hilbert space.

We assume that operators $A, B\colon V \to V'$, the functional $\varphi\colon V \times V \to \mathbb{R}$, and two initial values $\mathbf{u}_0 \in V, \mathbf{v}_0 \in H$.

There exists a constant $M_A > 0$ such that

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \geq M_A \|\mathbf{u}_1 - \mathbf{u}_2\|_V^2 \;, \forall \mathbf{u}_1, \mathbf{u}_2 \in V, \tag{4}$$

There exists a constant $L_A > 0$ such that

$$\|A\mathbf{u}_1 - A\mathbf{u}_2\|_{V'} \leq L_A \|\mathbf{u}_1 - \mathbf{u}_2\|_V \;, \forall \mathbf{u}_1, \mathbf{u}_2 \in V. \tag{5}$$

$B \in \mathcal{L}(V', V)$ is strongly monotone, i.e., there exists a constant $M_B > 0$ such that

$$(B(\mathbf{u}), \mathbf{u}) \geq M_B \|\mathbf{u}\|_V^2 \,, \forall \mathbf{u} \in V, \tag{6}$$

The norm of $B$ is $L_B > 0$, i.e.

$$\|B\mathbf{u}\|_{V'} \leq L_B \|\mathbf{u}\|_V \;, \forall \mathbf{u} \in V, \tag{7}$$

for any $\mathbf{u}, \mathbf{v} \in V$

$$(B\mathbf{u}, \mathbf{v}) = (B\mathbf{v}, \mathbf{u}), \tag{8}$$

There exists a constant $L_\varphi > 0$ such that

$$\varphi(\mathbf{g}_1, \mathbf{v}_2) + \varphi(\mathbf{g}_2, \mathbf{v}_1) - \varphi(\mathbf{g}_1, \mathbf{v}_1) - \varphi(\mathbf{g}_2, \mathbf{v}_2) \leq L_\varphi \|\mathbf{g}_1 - \mathbf{g}_2\| \|\mathbf{v}_1 - \mathbf{v}_2\|, \quad \forall \mathbf{g}_1, \mathbf{g}_2, \mathbf{v}_1, \mathbf{v}_2 \in V. \tag{9}$$

There exists a constant $C_\varphi > 0$ such that

$$\begin{aligned} &\varphi(\mathbf{g}, \mathbf{v}_1) - \varphi(\mathbf{g}, \mathbf{v}_2) \leq C_\varphi \|\mathbf{g}\|_V \|\mathbf{v}_1 - \mathbf{v}_2\|_V, \\ &\varphi(\mathbf{v}_1, \mathbf{g}) - \varphi(\mathbf{v}_2, \mathbf{g}) \leq C_\varphi \|\mathbf{g}\|_V \|\mathbf{v}_1 - \mathbf{v}_2\|_V, \quad \forall \mathbf{g}, \mathbf{v}_1, \mathbf{v}_2 \in V. \end{aligned} \tag{10}$$

For any $\mathbf{u} \in V$

$\varphi(\mathbf{u}, \cdot)$ is a convex functional in $V$. (11)

The function $\boldsymbol{f}$ satisfies

$$\boldsymbol{f} \in H^2(0, T; V'). \tag{12}$$

Then, if $L_\varphi \prec M_A \prec 2C_\varphi$. For each $\boldsymbol{f} \in V'$, the problem

$$\begin{cases} (\ddot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_{V' \times V} + (A\dot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_{V' \times V} + (B\mathbf{u}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_{V' \times V} + \varphi(\dot{\mathbf{u}}(t), \mathbf{v}) \\ \qquad -\varphi(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)) \geq (\boldsymbol{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_{V' \times V}, \forall \mathbf{v} \in V, \quad \text{a.e.} \;\; t \in [0, T], \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0. \end{cases}$$

has a unique solution

$$\mathbf{u}(t) \in C(0, T; V) \text{with} \begin{cases} \dot{\mathbf{u}}(t) \in C(0, T; V) \cap L^\infty(0, T; V), \\ and \\ \ddot{\mathbf{u}}(t) \in C(0, T; V) \cap L^\infty(0, T; V). \end{cases} \tag{13}$$

In the proof of the above Theorem 2, we employ the Rothe method to prove the existence and uniqueness of the solution, which may be found in Chen *et al*. (2020) page 06, Problem 3.1.

Finally, we recall the following standard result for parabolic variational inequalities (see Chen *et al*. (2020).)

Let $V \subset H \subset V'$ be a Gelfand triple. Let $K$ be a nonempty, closed, and convex set of $V$. Assume that $a(.,.)\colon V \times V \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\lambda > 0$ and $\gamma$,

$$a(\mathbf{v}, \mathbf{v}) = \gamma \parallel \mathbf{v} \parallel_H^2 \geq \lambda \parallel \mathbf{v} \parallel_V^2, \quad \forall \mathbf{v} \in H.$$

Then, for every $\mathbf{u}_0 \in K$ and $S \in L^2(0, T; H)$, there exists a unique function $\mathbf{u} \in H^1(0, T; H) \cap L^2(0, T; V)$, such that $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{u}(t) \in K$ for all $t \in [0, T]$, and for almost all $t \in (0, T)$,

$$(\dot{\mathbf{u}}(t), \mathbf{v} - \mathbf{u}(t))_{V' \times V} + a(\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t)) \geq (S(t), \mathbf{v} - \mathbf{u}(t))_H, \quad \forall \mathbf{v} \in K.$$

## 3. Mechanical and variational formulations

We give the physical setting of the contact problem and introduce some notations which we use in the sequel. We consider a viscoelastic body which occupies a domain $\Omega \subset \mathbb{R}^d$, where $d = 2,3$, such that the boundary $\Gamma = \partial\Omega$ is Lipschitz continuous.

As we mentioned earlier the boundary $\partial\Omega$ is divided into three disjoint measurable parts $\Gamma_1, \Gamma_2$ and $\Gamma_3$ with meas $(\Gamma_1) > 0$. In addition, we assume the body is in contact with a deformable foundation and the process is dynamic and frictional. We are interested in an evolution of the body in a finite time interval $(0,T)$.

We now have to write the boundary conditions on the contact surface $\Gamma_3$, We introduce the wear function $w: [0,T] \times \Gamma_3 \to \mathbb{R}^+$ which measures the wear of the surface. Wear is identified as the normal depth of material that is lost, the body is in bilateral contact with the foundation, as a result

$$u_\nu = -w \quad \text{on}\Gamma_3. \tag{14}$$

Thus, the location of the contact grows with wear. We recall that the effect of wear is on $\Gamma_3$ and therefore, it is natural to think that $u_\nu \leq 0$ on $\Gamma_3$, therefore $w > 0$ on $\Gamma_3$. The evolution of the wear of the contact surface is governed by a simplified version of Archard's law (See Strömberg, Johansson, and Klarbring (1996))

$$\dot{w} = -k\sigma_\nu \parallel \dot{\mathbf{u}}_\tau - \mathbf{v}^* \parallel, \tag{15}$$

where $k > 0$ is a coefficient of wear, $\mathbf{v}^*$ is the tangential velocity of the foundation and $\parallel \dot{\mathbf{u}}_\tau - \mathbf{v}^* \parallel$ represents the slip speed between the contact surface and the foundation. We assume that the motion of the foundation is uniform, i.e., $\mathbf{v}^*$ does not vary over time. We have Archard's law

$$\dot{w} = -kv^*\sigma_\nu. \tag{16}$$

The use of the simple law (16) avoids certain mathematical difficulties in the study of the dynamic problem of viscoelastic contact. Let $\zeta = kv^*$ and $\alpha = \frac{1}{\zeta}$ By using (14) and (16), we have

$$\sigma_\nu = \alpha\dot{u}_\nu. \tag{17}$$

We model the Coulomb dry friction contact between the body of the viscoelastic and the foundation as follows

$$\|\boldsymbol{\sigma}_\tau\| = \mu|\sigma_\nu|, \quad \boldsymbol{\sigma}_\tau = -\lambda(\dot{\mathbf{u}}_\tau - \mathbf{v}^*), \quad \lambda \geq 0, \tag{18}$$

where $\mu > 0$ is the coefficient of friction. Naturally, if $\dot{w} \geq 0$ Thus, it follows from (14) and (16) that $u_\nu \leq 0$ and $\sigma_\nu \leq 0$ on $\Gamma_3$ Thus, the conditions (17) and (18) imply

$$-\sigma_\nu = \alpha\|\dot{u}_\nu\|, \quad \|\boldsymbol{\sigma}_\tau\| = -\mu\sigma_\nu, \quad \boldsymbol{\sigma}_\tau = -\lambda(\dot{\mathbf{u}}_\tau - \mathbf{v}^*), \quad \lambda \geq 0. \tag{19}$$

The classical formulation of the mechanical problem of a frictional contact with wear may be stated as follows.

*problem **P***

Find a displacement field $\mathbf{u}: \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}: \Omega \times [0,T] \to \mathbb{S}^d$, a damage field $\beta: \Omega \times [0,T] \to \mathbb{R}$.

$$\rho\ddot{\mathbf{u}} = \text{Div}\,\boldsymbol{\sigma} + f_0 \quad \text{in}\Omega \times (0,T), \tag{20}$$

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}\varepsilon(\mathbf{u}(t)) + \int_0^t \mathcal{F}(t-s, \varepsilon(\mathbf{u}(s)), \beta(s))ds \quad \text{in}\Omega \times (0,T), \tag{21}$$

$$\dot{\beta} - k\Delta\beta + \partial\varphi_K(\beta) \ni \psi(\varepsilon(\mathbf{u}), \beta) \quad \text{in}\Omega \times (0,T), \tag{22}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on}\Gamma_1 \times (0,T), \tag{23}$$

$$\boldsymbol{\sigma\nu} = f_2 \quad \text{on}\Gamma_2 \times (0,T), \tag{24}$$

$$\begin{cases} -\sigma_\nu = \alpha \parallel \dot{u}_\nu \parallel, \\ \parallel \boldsymbol{\sigma}_\tau \parallel = -\mu\sigma_\nu, \\ \boldsymbol{\sigma}_\tau = -\lambda(\dot{\mathbf{u}}_\tau - v^*), \quad \lambda > 0 \end{cases} \quad \text{on} \quad \Gamma_3 \times (0,T), \tag{25}$$

$$\frac{\partial\beta}{\partial\nu} = 0 \quad \text{on}\Gamma \times (0,T), \tag{26}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \quad \beta(0) = \beta_0 \quad \text{in}\Omega, \tag{27}$$

The equation (20) represents the equation of motion where $f_0$ is the density of the volumic forces acting on the deformable body $\Omega$ and $\rho$ denotes the density of the mass. The equation (21) represents the constitutive law of a viscoelastic material with long-term memory where $\mathcal{F} = (\mathcal{F}_{ij})$ is the tensor of the relaxation and damage. The evolution of the damage field is modeled by the inclusion of the parabolic type given by relation (22) where $\psi$ is the source function of the damage, the set of admissible damage functions $K$ defined by

$$K = \{\xi \in H^1(\Omega): 0 \leq \xi \leq 1 \quad \text{a.e.in}\Omega\},$$

$\partial\varphi_K$ represents the subdifferential of the indicator function of set $K$. (23) - (24) are displacement and traction boundary conditions, respectively. The condition (25) describes the frictional bilateral contact with wear described above on the potential contact surface $\Gamma_3$. The relation (26) represents a homogeneous Neumann boundary condition where $\frac{\partial\beta}{\partial\nu}$ represents the normal derivative of $\beta$. In (27) we consider the initial conditions where $\mathbf{u}_0$ is the displacement initially, $\mathbf{v}_0$ the initial velocity field and $\beta_0$ the initial damage.

For the study of the mechanical problem (20) -(27) we consider the following hypotheses. The viscosity operator $\mathcal{A}: \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfied

$$\begin{cases} (a) \text{ There exists } L_{\mathcal{A}} > 0 \quad \text{such that} \\ \parallel \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2) \parallel \leq L_{\mathcal{A}} \parallel \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \parallel \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}_d, \text{a.e.}\,\mathbf{x} \in \Omega. \\ (b) \text{There exists } m_{\mathcal{A}} > 0 \quad \text{such that} \\ (\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)).(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \parallel \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \parallel^2 \\ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}_d, \text{a.e.}\,\mathbf{x} \in \Omega. \\ (c) \text{The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is lebesgue measurable on } \Omega, \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ (d) \text{The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \mathbf{0}) \in \mathcal{H}. \end{cases} \tag{28}$$

The elasticity operator $\mathcal{B}: \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfied

$$\begin{cases} (a) \text{There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ \parallel \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_2) \parallel \leq L_{\mathcal{B}} \parallel \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \parallel \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{a.e.}\,\mathbf{x} \in \Omega. \\ (b) \text{There exists} m_{\mathcal{B}} > 0 \quad \text{such that} \\ (\mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_2)).(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{B}} \parallel \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \parallel^2 \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{a.e.}\,\mathbf{x} \in \Omega. \\ (c) \quad \mathcal{B}\boldsymbol{\varepsilon}_1.\boldsymbol{\varepsilon}_2 = \mathcal{B}\varepsilon_2.\boldsymbol{\varepsilon}_1, \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{and} \mathcal{B}(\mathbf{0}) \in \mathcal{H}. \\ (d) \text{The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}) \text{ is lebesgue measurable on}\Omega, \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ (e) \text{The mapping } \mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \mathbf{0}) \in \mathcal{H}. \end{cases} \tag{29}$$

The relaxation tensor operator $\mathcal{F}: \Omega \times (0,T) \times \mathbb{S}^d \times \mathbb{R} \longrightarrow \mathbb{S}^d$ satisfied

$$\begin{cases} (a) \text{ There exists } L_{\mathcal{F}} > 0 \text{ such that} \\ \| \mathcal{F}(\mathbf{x}, t, \boldsymbol{\varepsilon}_1, \beta_1) - \mathcal{F}(\mathbf{x}, t, \boldsymbol{\varepsilon}_2, \beta_2) \| \leq L_{\mathcal{F}}(\| \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \| + \| \beta_1 - \beta_2 \|) \\ \qquad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \forall \beta_1, \beta_2 \in \mathbb{R}, \forall t \in (0,T), \text{a.e. } \mathbf{x} \in \Omega. \\ (b) \text{The mapping } \mathbf{x} \mapsto \mathcal{F}(x, t, \boldsymbol{\varepsilon}, \beta) \text{ is lebesgue measurable on } \Omega, \\ \qquad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \forall t \in (0,T), \beta \in \mathbb{R}. \\ (c) \text{The mapping } t \mapsto \mathcal{F}(\mathbf{x}, t, \boldsymbol{\varepsilon}, \beta) \text{ is continuous on } \Omega \times (0,T) \times \mathbb{S}^d \times \mathbb{R}. \\ (d) \text{The mapping } \mathbf{x} \mapsto \mathcal{F}(\mathbf{x}, t, \mathbf{0}, 0) \in \mathcal{H}, \forall t \in (0,T). \end{cases} \tag{30}$$

The function of the damage source $\psi: \Omega \times \mathbb{S}^d \times \mathbb{R} \longrightarrow \mathbb{R}$

$$\begin{cases} (a) \text{ There are } L_{\psi} > 0 \text{ such that} \\ \| \psi(\mathbf{x}, \boldsymbol{\varepsilon}_1, \beta_1) - \psi(\mathbf{x}, \boldsymbol{\varepsilon}_2, \beta_2) \| \leq L_{\psi}(\| \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \| + \| \beta_1 - \beta_2 \|) \\ \qquad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \forall \beta_1, \beta_2 \in \mathbb{R}, \text{a.e. } \mathbf{x} \in \Omega. \\ (b) \text{The mapping } \mathbf{x} \mapsto \psi(\mathbf{x}, \boldsymbol{\varepsilon}, \beta) \text{ is lebesgue measurable on } \Omega, \\ \qquad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \beta \in \mathbb{R}. \\ (c) \text{ The mapping } \mathbf{x} \mapsto \psi(\mathbf{x}, \mathbf{0}, 0) \in L^2(\Omega). \end{cases} \tag{31}$$

We assume that the function $\alpha$, the volume and surface forces satisfy

$$\rho \in L^\infty(\Gamma_3), \rho(\mathbf{x}) \geq \rho * > 0 \quad \text{a.e } \mathbf{x} \in \Omega. \tag{32}$$

$$\alpha \in L^\infty(\Gamma_3), \alpha(\mathbf{x}) \geq \alpha * > 0 \quad \text{a.e } \mathbf{x} \in \Gamma_3. \tag{33}$$

$$\boldsymbol{f}_0 \in H^2(0, T, L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in H^2(0, T, L^2(\Gamma_2)^d). \tag{34}$$

The coefficient of friction $\mu$ is such that

$$\mu \in L^\infty(\Gamma_3), \mu(\mathbf{x}) > 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_3. \tag{35}$$

Finally the initial conditions satisfied

$$\mathbf{u}_0 \in V, \quad \mathbf{v}_0 \in H. \tag{36}$$

$$\beta_0 \in K. \tag{37}$$

We define the bilinear form $a: H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$

$$a(\xi, \zeta) = k \int_\Omega \nabla\xi \nabla\zeta dx, \tag{38}$$

where $k$ is a positive coefficient.

Define an inner product $(.,.)_H$ by setting

$$(\mathbf{u}, \mathbf{v})_H = (\rho(\mathbf{x})\mathbf{u}, \mathbf{v})_H.$$

Riesz's representation Theorem causes the existence of an element $f \in V'$, such that

$$(\boldsymbol{f}(t), \mathbf{v})_{V' \times V} = (\boldsymbol{f}_0(t), \mathbf{v})_H + (\boldsymbol{f}_2(t), \mathbf{v})_{L^2(\Omega)}, \forall \mathbf{v} \in V \quad \text{a.e. } t \in (0,T). \tag{39}$$

Note that condition (34) implies that

$$\boldsymbol{f} \in H^2(0, T; V'). \tag{40}$$

Now let $\varphi: V \times V \to \mathbb{R}$, the mapping defined by

$$\varphi(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \alpha \|u_\nu\| (\mu \|\mathbf{v} - \mathbf{v}^*\| + v_\nu) \, da. \tag{41}$$

Using standard arguments based on Green's formula for example (see Chen *et al*. (2020), page 4-5), we obtain the variational formulation of the problem (20)-(27) is given by *problem **PV***

$$\boldsymbol{\sigma}(t)=\mathcal{A}\varepsilon\big(\dot{\mathbf{u}}(t)\big)+\mathcal{B}\varepsilon\big(\mathbf{u}(t)\big)+\int_0^t \mathcal{F}\left(t-s,\varepsilon\big(\mathbf{u}(s)\big),\beta(s)\right)ds,\quad t\in(0,T), \tag{42}$$

$$\begin{aligned}(\ddot{\mathbf{u}}(t),\mathbf{v}-\dot{\mathbf{u}}(t))_{V'\times V}\quad &+(\boldsymbol{\sigma}(t),\varepsilon(\mathbf{v}-\dot{\mathbf{u}}(t)))_{\mathcal{H}}+\varphi(\dot{\mathbf{u}}(t),\mathbf{v})-\varphi(\dot{\mathbf{u}}(t),\dot{\mathbf{u}}(t))\\ &\geq(\boldsymbol{f}(t),(\mathbf{v}-\dot{\mathbf{u}}(t)))_{V'\times V},\quad \forall\mathbf{v}\in V,\ t\in[0,T],\end{aligned} \tag{43}$$

$$\begin{aligned}\beta(t)\in K,\qquad &(\dot{\beta}(t),\zeta-\beta(t))_{L^2(\Omega)}+a(\beta(t),\zeta-\beta(t))\\ &\geq(\psi(\varepsilon(\mathbf{u}(t)),\beta(t)),\zeta-\beta(t))_{L^2(\Omega)},\quad \forall\zeta\in K,\ t\in[0,T],\end{aligned} \tag{44}$$

$$\mathbf{u}(0)=\mathbf{u}_0,\quad \dot{\mathbf{u}}(0)=\mathbf{v}_0,\quad \beta(0)=\beta_0. \tag{45}$$

Our main existence and uniqueness result for Problem $PV$ is the following.

## 4. Existence and uniqueness

Assume that (28)-(37) hold, Then there exists a constant $\lambda_0>0$ depending on $\Gamma_3$, such that if

$$\|\alpha\|_{L^\infty(\Gamma_3)}\left(\|\mu\|_{L^\infty(\Gamma_3)}+1\right)<\lambda_0. \tag{46}$$

and

$$c_\gamma^2\|\alpha\|_{L^\infty(\Gamma_3)^d}\left(\|\mu\|_{L^\infty(\Gamma_3)^d}+1\right)<m_{\mathcal{A}}<2c_\gamma^2\|\alpha\|_{L^\infty(\Gamma_3)^d}\left(\|\mu\|_{L^\infty(\Gamma_3)^d}+1\right), \tag{47}$$

where $c_\gamma$ is a constant. Then Problem $PV$ has a unique solution $(\mathbf{u},\boldsymbol{\sigma},\beta)$. Moreover, the solution satisfies

$$\begin{cases}\mathbf{u}(t)\in C(0,T;V)\\ \dot{\mathbf{u}}(t)\in C(0,T;V)\cap L^\infty(0,T;V)\\ \ddot{\mathbf{u}}(t)\in C(0,T;V)\cap L^\infty(0,T;V).\end{cases} \tag{48}$$

$$\boldsymbol{\sigma}\in H^2(0,T;\mathcal{H}),\quad Div\boldsymbol{\sigma}\in H^2(0,T;V'), \tag{49}$$

$$\beta\in W^{1,2}(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)), \tag{50}$$

Note that if $\mathbf{v}^*$ is large enough, then $\alpha=\frac{1}{k\mathbf{v}^*}$ is sufficiently small and, therefore, the condition (46) for the unique solution of the $PV$ problem is satisfied. We conclude that the mechanical problem (20)-(27) has a unique weak solution if the tangential velocity of the foundation is large enough. In addition, the solving the problem (20)-(27), allows us to find the function of the wear by integration of (16) and we use the initial condition $w(0)=0$ which indicates that the body at the initial time is not subjected to wear.

The proof of Theorem 4, is carried out is several steps and is based on the following abstract result for evolutionary hyperbolic quasi-variational inequality. We denote by $C$ a constant whose value may change from line to line when no confusing can arise.

Let $\boldsymbol{\eta}\in H^2(0,T;V')$ be given and consider the following variational problems.

*Problem* $\boldsymbol{\mathcal{P}_\eta}$

Find a displacement field $\mathbf{u}_\eta:[0,T]\to V$, such that

$$\begin{aligned}&(\ddot{\mathbf{u}}_\eta(t),\mathbf{v}-\dot{\mathbf{u}}_\eta(t))_{V'\times V}+\big(\mathcal{A}\varepsilon(\dot{\mathbf{u}}_\eta(t)),\varepsilon(\mathbf{v}-\dot{\mathbf{u}}_\eta(t))\big)_{\mathcal{H}}+\big(\mathcal{B}\varepsilon(\mathbf{u}_\eta(t)),\varepsilon(\mathbf{v}-\dot{\mathbf{u}}_\eta(t))\big)_{\mathcal{H}}\\ &+\big(\boldsymbol{\eta}(t),\mathbf{v}-\dot{\mathbf{u}}_\eta(t)\big)_{V'\times V}+\varphi\big(\dot{\mathbf{u}}_\eta(t),\mathbf{v}\big)-\varphi\big(\dot{\mathbf{u}}_\eta(t),\dot{\mathbf{u}}_\eta(t)\big)\geq\big(\boldsymbol{f}(t),\mathbf{v}-\dot{\mathbf{u}}_\eta(t)\big)_{V'\times V}\end{aligned} \tag{51}$$

a.e. $t\in(0,T)$, for all $\mathbf{v}\in V$,

$$\mathbf{u}_\eta(0)=\mathbf{u}_0,\quad \dot{\mathbf{u}}_\eta(0)=\dot{\mathbf{u}}_0=\mathbf{v}_0. \tag{52}$$

We have the following result for $\mathcal{P}_\eta$ There exists $\mathbf{u}_\eta$ a unique solution to Problem $\mathcal{P}_\eta$ and it has the regularity expressed in (48).

*Proof.* We apply Theorem 2. Thus, we only need to verify that all the conditions (4)-(12) are satisfied. The Riesz representation Theorem allows us to define $\boldsymbol{f}_\eta:[0,T]\to V$, by

$$\big(\boldsymbol{f}_\eta(t),\mathbf{v}\big)_{V'\times V}=(\boldsymbol{f}(t)-\boldsymbol{\eta}(t),\mathbf{v})_{V'\times V}.$$

We define the operator $A:V\to V'$

$$(A\mathbf{u},\mathbf{v})_{V'\times V}=(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}(t)),\varepsilon(\mathbf{v}))_{\mathcal{H}}\quad \forall\mathbf{u},\mathbf{v}\in V, \tag{53}$$

it follows from (53) and (28)(a) that

$$\| A\mathbf{u}_1 - A\mathbf{u}_2 \|_{V'} = \| \mathcal{A}\varepsilon(\mathbf{u}_1) - \mathcal{A}\varepsilon(\mathbf{u}_2) \|_{\mathcal{H}} \leq L_{\mathcal{A}} \| \varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2) \|_{\mathcal{H}}$$
$$\leq L_{\mathcal{A}} \| \mathbf{u}_1 - \mathbf{u}_2 \|_V.$$

Now by(53) and (28)(b), we find

$$(A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_{V'\times V} = (\mathcal{A}\varepsilon(\mathbf{u}_1) - \mathcal{A}\varepsilon(\mathbf{u}_2), \varepsilon(\mathbf{u}_1 - \mathbf{u}_2))_{\mathcal{H}}$$
$$\geq m_{\mathcal{A}} \| \varepsilon(\mathbf{u}_1 - \mathbf{u}_2)) \|_{\mathcal{H}}^2$$
$$\geq m_{\mathcal{A}} \| \mathbf{u}_1 - \mathbf{u}_2) \|_V^2.$$

Similarly for $B: V \to V'$

$$(B\mathbf{u}, \mathbf{v})_{V'\times V} = (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in V, \tag{54}$$

it follows from (54) and (29)(a) that

$$\| B\mathbf{u}_1 - B\mathbf{u}_2 \|_{V'} = \| \mathcal{B}\varepsilon(\mathbf{u}_1) - \mathcal{B}\varepsilon(\mathbf{u}_2) \|_{\mathcal{H}} \leq L_{\mathcal{B}} \| \varepsilon(\mathbf{u}_1) - \varepsilon(\mathbf{u}_2) \|_{\mathcal{H}}$$
$$\leq L_{\mathcal{B}} \| \mathbf{u}_1 - \mathbf{u}_2 \|_V.$$

Now by (54) and (29)(b), we find

$$(B\mathbf{u}_1 - B\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_{V'\times V} = (\mathcal{B}\varepsilon(\mathbf{u}_1) - \mathcal{B}\varepsilon(\mathbf{u}_2), \varepsilon(\mathbf{u}_1 - \mathbf{u}_2))_{\mathcal{H}}$$
$$\geq m_{\mathcal{B}} \| \varepsilon(\mathbf{u}_1 - \mathbf{u}_2)) \|_{\mathcal{H}}^2$$
$$\geq m_{\mathcal{B}} \| \mathbf{u}_1 - \mathbf{u}_2) \|_V^2.$$

For the condition (8), it results directly from (29)(c).
Clearly, (4)-(8) and (12) are met. Now we turn to check the remaining conditions. Since

$$\varphi(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \alpha \|u_\nu\| (\mu \|v - v^*\| + v_\nu) da.$$

and for any $\mathbf{v}_1, \mathbf{v}_2 \in V$

$$|\lambda \mathbf{v}_1 + (1-\lambda)\mathbf{v}_2 - \mathbf{v}^*| \leq \lambda |\mathbf{v}_1 - \mathbf{v}^*| + (1-\lambda)|\mathbf{v}_2 - \mathbf{v}^*|,$$

we deduce that $\varphi(\mathbf{u}, .)$ is a proper convex functional

$$\begin{aligned}
\varphi(\boldsymbol{g}, \mathbf{v}_1) - \varphi(\boldsymbol{g}, \mathbf{v}_2) &= \int_{\Gamma_3} \alpha \|g_\nu\| \left(\mu \left\|\mathbf{v}_{1,\tau} - \mathbf{v}^*\right\| + v_{1,\nu}\right) da \\
&\quad - \int_{\Gamma_3} \alpha \|g_\nu\| \left(\mu \left\|\mathbf{v}_{2,\tau} - \mathbf{v}^*\right\| + v_{2,\nu}\right) da \\
&= \int_{\Gamma_3} \alpha \|g_\nu\| \left(\mu \left\|\mathbf{v}_{1,\tau} - \mathbf{v}^*\right\| - \mu \left\|\mathbf{v}_{2,\tau} - v^*\right\| + v_{1,\nu} - v_{2,\nu}\right) da \\
&\leq \int_{\Gamma_3} \alpha \|g_\nu\| \left(\mu \left\|\mathbf{v}_{1,\tau} - \mathbf{v}_{2,\tau}\right\| + \left\|v_{1,\nu} - v_{2,\nu}\right\|\right) d\Gamma \\
&\leq \| \alpha \|_{L^\infty(\Gamma_3)} \| \mu \|_{L^\infty(\Gamma_3)} \|g_\nu\|_{L^2(\Gamma_3)} \left\|\mathbf{v}_{1,\tau} - \mathbf{v}_{2,\tau}\right\|_{L^2(\Gamma_3)} \\
&\quad + \| \alpha \|_{L^\infty(\Gamma_3)} \|g_\nu\|_{L^2(\Gamma_3)} \left\|v_{1,\nu} - v_{2,\nu}\right\|_{L^2(\Gamma_3)} \\
&\leq \| \alpha \|_{L^\infty(\Gamma_3)} \left(\| \mu \|_{L^\infty(\Gamma_3)} + 1\right) \|g_\nu\|_{L^2(\Gamma_3)} \left\|v_{1,\nu} - v_{2,\nu}\right\|_{L^2(\Gamma_3)} \\
&\leq \| \alpha \|_{L^\infty(\Gamma_3)} \left(\| \mu \|_{L^\infty(\Gamma_3)} + 1\right) \| g \|_{L^2(\Gamma_3)} \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^2(\Gamma_3)}.
\end{aligned} \tag{55}$$

We know that there exists a constant $c_\gamma > 0$ such that

$$\| \mathbf{u} \|_{L^2(\Gamma_3)^d} \leq \| \mathbf{u} \|_{L^2(\Omega)^d} \leq c_\gamma \| \mathbf{u} \|_V, \quad \forall \mathbf{u} \in L^2(\Gamma_3)^d.$$

Then the inequality (55) can be transformed as follows

$$\varphi(\mathbf{g}, \mathbf{v}_1) - \varphi(\mathbf{g}, \mathbf{v}_2) \leq c_\gamma^2 \| \alpha \|_{L^\infty(\Gamma_3)} \left(\| \mu \|_{L^\infty(\Gamma_3)} + 1\right) \| \mathbf{g} \|_V \|\mathbf{v}_1 - \mathbf{v}_2\|_V.$$

Thus the condition (10) holds. Similarly, we have

$$\varphi(g_1, v_2) - \varphi(g_1, v_1) + \varphi(g_2, v_1) - \varphi(g_2, v_2) \leq$$
$$c_\gamma^2 \| \alpha \|_{L^\infty(\Gamma_3)} \left(\left(\mu_{L^\infty(\Gamma_3)} + 1\right) \|g_1 - g_2\|_V \|v_1 - v_2\|_V.\right.$$

and so the condition (9) is true, and we point out that (47) has been achieved $L_\varphi \prec M_A \prec 2C_\varphi$. Therefore, we verify that all the conditions of Theorem 2 are satisfied and so Problem $\mathcal{P}_\eta$ is uniquely solvable.
For $\theta \in H^2(0, T; L^2(\Omega))$, we consider the following variational problem.
*Problem* $\boldsymbol{\mathcal{P}_\theta}$

Find the damage field $\beta_\theta : [0,T] \to \mathbb{R}$, such that

$$\beta_\theta(t) \in K, \left(\dot{\beta}_\theta(t), \xi - \beta_\theta(t)\right)_{L^2(\Omega)} + a(\beta_\theta(t), \xi - \beta_\theta(t)) \geq (\theta(t), \xi - \beta_\theta(t))_{L^2(\Omega)}, \quad \forall \xi \in K, \text{a.e. t} \in (0,T), \tag{56}$$

$$\beta_\theta(0) = \beta_0. \tag{57}$$

There exists a unique solution $\beta_\theta$ to the auxiliary problem $\mathcal{P}_\theta$ satisfying (50).

*Proof.* The inclusion mapping of $\left(H^1(\Omega), \|\cdot\|_{H^1(\Omega)}\right)$ into $\left(L^2(\Omega), \|\cdot\|_{L^2(\Omega)}\right)$ is continuous and its range is dense. We denote by $(H^1(\Omega))'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $L^2(\Omega)$ with itself, we can write the Gelfand triple

$$H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))'.$$

We use the notation $(.,.)_{(H^1(\Omega))' \times H^1(\Omega)}$ to represent the duality pairing between $(H^1(\Omega))'$ and $(H^1(\Omega))$. We have

$$(\beta, \xi)_{(H^1(\Omega))' \times H^1(\Omega)} = (\beta, \xi)_{L^2(\Omega)}, \quad \forall \beta \in L^2(\Omega), \xi \in H^1(\Omega),$$

and we note that $K$ is a closed convex set in $(H^1(\Omega))$. Then, using the definition (38) of the bilinear form $a$ , and the fact that $\beta_\theta \in K$ in (37), it is easy to see that Lemma 4 is a consequence of Theorem 2.

We now consider the operator

$$\Lambda: H^2\left(0,T; V' \times L^2(\Omega)\right) \to H^2\left(0,T; V' \times L^2(\Omega)\right),$$

defined by

$$\Lambda(\boldsymbol{\eta}, \theta)(t) = \left(\Lambda^1(\boldsymbol{\eta}, \theta)(t), \Lambda^2(\boldsymbol{\eta}, \theta)(t)\right) \in V' \times L^2(\Omega), \tag{58}$$

and

$$(\Lambda^1(\boldsymbol{\eta}, \theta)(t), \mathbf{v})_{V' \times V} = \left(\int_0^t \mathcal{F}\left(t-s, \varepsilon\left(\mathbf{u}_{\boldsymbol{\eta}}(s)\right), \beta_\theta(s)\right) ds, \varepsilon(\mathbf{v})\right)_{\mathcal{H}}, \forall \mathbf{v} \in V, \tag{59}$$

$$\Lambda^2(\boldsymbol{\eta}, \theta)(t) = \psi\left(\mathbf{u}_\eta(t), \beta_\theta(t)\right). \tag{60}$$

We have the following result. The mapping $\Lambda$ has a fixed point $(\boldsymbol{\eta}^*, \theta^*) \in H^2\left(0,T; V' \times L^2(\Omega)\right)$. Such that $\Lambda(\boldsymbol{\eta}^*, \theta^*) = (\boldsymbol{\eta}^*, \theta^*)$.

*Proof.* Let $(\boldsymbol{\eta}_1, \theta_1), (\boldsymbol{\eta}_2, \theta_2) \in H^2\left(0,T; V' \times L^2(\Omega)\right)$. We use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\dot{\mathbf{u}}_{\eta_i} = \dot{\mathbf{u}}_i$, $\ddot{\mathbf{u}}_{\eta_i} = \ddot{\mathbf{u}}_i$ and $\beta_{\theta_i} = \beta_i$ , for $i = 1,2$.

using(30), we have

$$\|\Lambda^1(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda^1(\boldsymbol{\eta}_2, \theta_2)(t)\|_V^2, \quad \leq C\left(\int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 ds + \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Omega)}^2 ds\right), \tag{61}$$

and by (31) we find

$$\|\Lambda^2(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda^2(\boldsymbol{\eta}_2, \theta_2)(t)\|_{L^2(\Omega)}^2 \quad \leq C\left(\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|\beta_1(s) - \beta_2(s)\|_{L^2(\Omega)}^2\right), \tag{62}$$

Therefore,

$$\|\Lambda(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda(\boldsymbol{\eta}_2, \theta_2)(t)\|_{V' \times L^2(\Omega)}^2 \leq C(\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 ds + \|\beta_1(s) - \beta_2(s)\|_{L^2(\Omega)}^2 + \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Omega)}^2 ds\Big). \tag{63}$$

Using inequality (51) for $\boldsymbol{\eta} = \boldsymbol{\eta}_1$ we find

$$(\ddot{\mathbf{u}}_1, \mathbf{v} - \dot{\mathbf{u}}_1)_{V' \times V} + (\mathcal{A}\varepsilon(\dot{\mathbf{u}}_1), \varepsilon(\mathbf{v} - \dot{\mathbf{u}}_1))_{\mathcal{H}} + (\mathcal{B}\varepsilon(\mathbf{u}_1), \varepsilon(\mathbf{v} - \dot{\mathbf{u}}_1))_{\mathcal{H}} + (\boldsymbol{\eta}_1, \mathbf{v} - \dot{\mathbf{u}}_1)_{V' \times V} + \varphi(\dot{\mathbf{u}}_1, \mathbf{v}) - \varphi(\dot{\mathbf{u}}_1, \dot{\mathbf{u}}_1) \geq (\boldsymbol{f}, \mathbf{v} - \dot{\mathbf{u}}_1)_{V' \times V}, \tag{64}$$

Also for $\boldsymbol{\eta} = \boldsymbol{\eta}_2$ we find

$$(\ddot{\mathbf{u}}_2, \mathbf{v} - \dot{\mathbf{u}}_2)_{V' \times V} + (\mathcal{A}\varepsilon(\dot{\mathbf{u}}_2), \varepsilon(\mathbf{v} - \dot{\mathbf{u}}_2))_{\mathcal{H}} + (\mathcal{B}\varepsilon(\mathbf{u}_2), \varepsilon(\mathbf{v} - \dot{\mathbf{u}}_2))_{\mathcal{H}} + (\boldsymbol{\eta}_2, \mathbf{v} - \dot{\mathbf{u}}_2)_{V' \times V} + \varphi(\dot{\mathbf{u}}_2, \mathbf{v}) - \varphi(\dot{\mathbf{u}}_2, \dot{\mathbf{u}}_2) \geq (\boldsymbol{f}, \mathbf{v} - \dot{\mathbf{u}}_2)_{V' \times V}, \tag{65}$$

we take $\mathbf{v} = \dot{\mathbf{u}}_2$ in (64) and $\mathbf{v} = \dot{\mathbf{u}}_1$ in (65) by adding the results obtained we have

$$(\ddot{\mathbf{u}}_1-\ddot{\mathbf{u}}_2,\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2)_{V'\times V}+(\mathcal{A}\varepsilon(\dot{\mathbf{u}}_1)-\mathcal{A}\varepsilon(\dot{\mathbf{u}}_2),\varepsilon(\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2))_{\mathcal{H}}$$
$$+(\mathcal{B}\varepsilon(\mathbf{u}_1)-\mathcal{B}\varepsilon(\mathbf{u}_2),\varepsilon(\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2))_{\mathcal{H}}+(\boldsymbol{\eta}_1-\boldsymbol{\eta}_2,\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2)_{V'\times V}$$
$$\leq\varphi(\dot{\mathbf{u}}_1,\dot{\mathbf{u}}_2)-\varphi(\dot{\mathbf{u}}_1,\dot{\mathbf{u}}_1)+\varphi(\dot{\mathbf{u}}_2,\dot{\mathbf{u}}_1)-\varphi(\dot{\mathbf{u}}_2,\dot{\mathbf{u}}_2),$$

Combine this hypothesis with (28)-(29) and the inequalities provided by the functional $\varphi$ we find

$$\frac{1}{2}\frac{d}{dt}\|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\|_V^2+m_{\mathcal{A}}\|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\|_V^2\leq L_{\mathcal{B}}\|\mathbf{u}_1-\mathbf{u}_2\|_V\|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\|_V$$
$$+\|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2\|_V\|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\|_V+c_\gamma^2\parallel\alpha\parallel_{L^\infty(\Gamma_3)^d}\left(\parallel\mu\parallel_{L^\infty(\Gamma_3)^d}+1\right)\|\dot{\mathbf{u}}_1-\dot{\mathbf{u}}_2\|_V^2, \quad (66)$$

we integrate this inequality with respect to time and by Young's inequality and by to use $\mathbf{u}_i(t)=\int_0^t\dot{\mathbf{u}}_i(s)ds+\mathbf{u}_0$, and by from Gronwall, we find

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V^2\leq C\int_0^t\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_V^2ds. \quad (67)$$

From (56), deduced that

$$\left(\dot{\beta}_1-\dot{\beta}_2,\beta_1-\beta_2\right)_{L^2(\Omega)}+a(\beta_1-\beta_2,\beta_1-\beta_2)\leq(\theta_1-\theta_2,\beta_1-\beta_2)_{L^2(\Omega)},\forall t\in(0,T),$$

integrate inequality with respect to time, using the initial condyions $\beta_1(0)=\beta_2(0)=\beta_0$, and inequality $a(\beta_1-\beta_2,\beta_1-\beta_2)\geq 0$, we find

$$\frac{1}{2}\|\beta_1(t)-\beta_2(t)\|_{L^2(\Omega)}^2\leq C\int_0^t\left(\theta_1(s)-\theta_2(s),\beta_1(s)-\beta_2(s)\right)_{L^2(\Omega)}ds,$$

which implies

$$\|\beta_1(t)-\beta_2(t)\|_{L^2(\Omega)}^2\leq C\left(\int_0^t\|\theta_1(s)-\theta_2(s)\|_{L^2(\Omega)}^2ds+\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Omega)}^2ds\right), \quad (68)$$

this inequality combined with the Gronwall inequality leads to

$$\|\beta_1(t)-\beta_2(t)\|_{L^2(\Omega)}^2\leq C\int_0^t\|\theta_1(s)-\theta_2(s)\|_{L^2(\Omega)}^2ds,\quad\forall t\in[0,T]. \quad (69)$$

Using (63), (67) and (69), we find

$$\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|_{V'\times L^2(\Omega)}^2\leq C\int_0^t\|(\boldsymbol{\eta}_1,\theta_1)(s)-(\boldsymbol{\eta}_2,\theta_2)(s)\|_{V'\times L^2(\Omega)}^2ds. \quad (70)$$

Reiterant cette inegalite $m$ lead time a

$$\|\Lambda^m(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda^m(\boldsymbol{\eta}_2,\theta_2)(t)\|_{H^2\left(0,T;V'\times L^2(\Omega)\right)}^2$$
$$\leq\frac{(CT)^m}{m!}\|(\boldsymbol{\eta}_1,\theta_1)-(\boldsymbol{\eta}_2,\theta_2)\|_{H^2\left(0,T;V'\times L^2(\Omega)\right)}^2,$$

this inequality shows that for a sufficiently large the operator $\Lambda^m$ is a contraction operator in the Banach space $H^2\left(0,T;V'\times L^2(\Omega)\right)$. Therefor, there exists a unique element $(\boldsymbol{\eta}^*,\theta^*)\in H^2\left(0,T;V'\times L^2(\Omega)\right)$, such that

$$\Lambda(\boldsymbol{\eta}^*,\theta^*)=(\boldsymbol{\eta}^*,\theta^*).$$

Now we have every thing that is required to prove Theorem 4.

**Existence**

Let $(\boldsymbol{\eta}^*,\theta^*)\in H^2\left(0,T;V'\times L^2(\Omega)\right)$, be the fixed point of $\Lambda$ and denote

$$\mathbf{u}_*=\mathbf{u}_{\boldsymbol{\eta}^*},\beta_*=\beta_{\theta^*}, \quad (71)$$

$$\boldsymbol{\sigma}_*=\mathcal{A}\varepsilon(\dot{\mathbf{u}}_*)+\mathcal{B}\varepsilon(\mathbf{u}_*)+\int_0^t\mathcal{F}(t-s,\varepsilon(\mathbf{u}_*(s)),\beta_*(s))ds,\quad t\in(0,T), \quad (72)$$

we use : $\Lambda^1(\boldsymbol{\eta}^*,\theta^*)=\boldsymbol{\eta}^*$, $\Lambda^2(\boldsymbol{\eta}^*,\theta^*)=\theta^*$, it follows

$$(\boldsymbol{\eta}^*(t),\mathbf{v})_{V'\times V}=\left(\int_0^t\mathcal{F}(t-s,\varepsilon(\mathbf{u}_*(s)),\beta_*(s))ds,\varepsilon(\mathbf{v})\right)_{\mathcal{H}},\forall\mathbf{v}\in V. \quad (73)$$

$$\theta^*(t)=S\left(\mathbf{u}_*(t),\beta_*(t)\right). \quad (74)$$

We prove $(\mathbf{u}_*,\boldsymbol{\sigma}_*,\beta_*)$ satisfies (42)-(45) and the regularities (48)-(50). Indeed, we write (51) for $\boldsymbol{\eta}=\boldsymbol{\eta}^*$ and use (71) to find

$$(\ddot{\mathbf{u}}_*(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))_{V'\times V}+(\mathcal{A}\varepsilon(\dot{\mathbf{u}}_*)(t),\varepsilon(\mathbf{v}-\dot{\mathbf{u}}_*(t)))_{\mathcal{H}}+(\mathcal{B}\varepsilon(\mathbf{u}_*(t)),\varepsilon(\mathbf{v}-\dot{\mathbf{u}}_*(t)))_{\mathcal{H}}$$
$$+(\boldsymbol{\eta}^*(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))_{V'\times V}+\varphi(\dot{\mathbf{u}}_*(t),\mathbf{v})-\varphi(\dot{\mathbf{u}}_*(t),\dot{\mathbf{u}}_*(t))\geq(\boldsymbol{f}(t),\mathbf{v}-\dot{\mathbf{u}}_*(t))_{V'\times V}, \quad (75)$$
$$\text{a.e.}\quad t\in(0,T),\text{for all }\mathbf{v}\in V.$$

Substitute (73) in (75) to obtain

$$(\ddot{\mathbf{u}}_*(t), \mathbf{v}-\dot{\mathbf{u}}_*(t))_{V'\times V} + (\mathcal{A}\varepsilon(\dot{\mathbf{u}}_*)(t), \varepsilon(\mathbf{v}-\dot{\mathbf{u}}_*(t)))_{\mathcal{H}} + (\mathcal{B}\varepsilon(\mathbf{u}_*(t)), \varepsilon(\mathbf{v}-\dot{\mathbf{u}}_*(t)))_{\mathcal{H}} + \left(\int_0^t \mathcal{F}(t-s, \varepsilon(\mathbf{u}_*(s)), \beta_*(s))ds, \varepsilon(\mathbf{v}-\dot{\mathbf{u}}_*(t))\right)_{\mathcal{H}} + \varphi(\dot{\mathbf{u}}_*(t), \mathbf{v}) - \varphi(\dot{\mathbf{u}}_*(t), \dot{\mathbf{u}}_*(t)) \geq \left(\boldsymbol{f}(t), \mathbf{v}-\dot{\mathbf{u}}_*(t)\right)_{V'\times V'}, \quad \text{a.e. } t\in(0,T), \text{for all } \mathbf{v}\in V. \tag{76}$$

and we write (56) for $\theta = \theta^*$ and use (71) to find

$$\beta_*(t)\in K, \left(\dot{\beta}_*(t), \xi-\beta_*(t)\right)_{L^2(\Omega)} + a(\beta_*(t), \xi-\beta_*(t)) \geq (\theta^*(t), \xi-\beta_*(t))_{L^2(\Omega)}, \forall\xi\in K, \text{a.e. t}\in(0,T), \tag{77}$$

we substitute (74) in (77) to obtain

$$\beta_*(t)\in K, \left(\dot{\beta}_*(t), \xi-\beta_*(t)\right)_{L^2(\Omega)} + a(\beta_*(t), \xi-\beta_*(t)) \geq \left(S\big(\mathbf{u}_*(t), \beta_*(t)\big), \xi-\beta_*(t)\right)_{L^2(\Omega)}, \forall\xi\in K, \ \text{a.e.} \quad \text{t}\in(0,T). \tag{78}$$

The relations (75)-(78), allow us to conclude now that $(\mathbf{u}_*, \boldsymbol{\sigma}_*, \beta_*)$ satisfies (42)-(44). Next, (45) the regularity (48)-(50) follow from Lemmas 4 and 4. Since $\mathbf{u}_*, \beta_*$ satisfies (48), (50), respectively, It follows from (34) that

$$\boldsymbol{\sigma}_* \in H^2(0,T;V') \tag{79}$$

we choose $\mathbf{v} = \mathbf{u} \pm \phi$ in (76), with $\phi \in D(\Omega)^d$ to obtain

$$Div\, \boldsymbol{\sigma}_*(t) = f_0(t), \forall t\in[0,T], \tag{80}$$

where $D(\Omega)$ is the space of infinitely differentiable real functions with a compact support in $\Omega$. The regularity (49) follows from (34), (79) and (80).

Finally we conclude that the weak solution $(\mathbf{u}_*, \boldsymbol{\sigma}_*, \beta_*)$ of the problem $PV$ has the regularity (48)–(50), which concludes the existence part of Theorem 4.

***Uniqueness***

The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator

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