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Extropy estimation of Weibull distribution under upper records

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Nour El Houda Zouaoui
ORCID: https://orcid.org/0000-0003-3224-8273
Laboratory of Applied Mathematics, Mathematics Department, Biskra University, Algeria
E-mail: nourelhouda.zouaoui@univ-biskra.dz, nor9180327@ju.edu.jo

Abstract
Recently, extropy has gained interest by the academic researchers. This work explores the features of parametric and non-parametric estimators based on upper record values under the a two-parameter Weibull distribution. We apply the Markov Chain Monte Carlo (MCMC) method to provide a Bayesian estimator. A considerable number of theoretical properties of the procedures are determined.
Keywords: Extropy. Upper record. Bayesian estimation

1. Introduction

In reliability theory, the concept of uncertainty plays a pivotal role in our understanding of data and decision-making processes. Uncertainty, often quantified by entropy, see Shannon (1984), measures the degree to which outcomes are unpredictable or unknown. In the other hand, an alternative measurement for uncertainty was suggested by Lad *et al*. (2015) named extropy. offering a different perspective on the information contained within a system. For a continuous random variable (rv) X with with probability density function (pdf) $f\ (x)$ and cumulative distribution function (cdf) $F(x)$, the extropy is defined by

$$J(X) = -\frac{1}{2}\int f^2(x)dx \qquad (1)$$

These measurements are highly regarded in the framework of order and record statistics. In this article, we adopt the concept of record values which is important in a variety of practical applications, ranging from reliability engineering to environmental research. Understanding the behavior and characteristics of these records can provide useful information about the underlying distribution and its properties. readers may refer to works such as Arnold *et al*. (1998).

Let $X_1, \ldots, X_n$ be a sequence of independent and identically distributed (iid) rv with pdf $f\ (x)$ and cdf $F(x)$. We say an upper record value is an observation X_j that outperforms all prior observations in a series of random variables $X_j > X_i, \forall\ i < j$ (Ahsanullah, 1995).

Assume $D_n = U_1, \ \ldots, U_n$ be the first n upper record values. The joint density function for D_n is as follows:

$$f(u_1, \ldots, u_n) = f(u_n)\prod_{i=1}^{n-1}\frac{f(u_i)}{1-f(u_i)} \qquad (2)$$

The estimation of extropy has recently attracted attention from several researchers. We may refer to Qiu *et al*. (2017), who provided some estimators for extropy with applications in testing uniformity. On the other hand, the problem of estimating the extropy based on complete sample has been considered recently by some authors, see for example, Qiu *et al*. (2018). Characterization results are also given. Additionally, Zouaoui *et al*. (2022) investigated the Evaluation of the uncertainty increments for the records and they also provided related characterization results. For more statistical inference see Jose *et al*. (2019) and Ahmed *et al*. (2023).

This paper aims to explore the extropy of record values within the context of the Weibull distribution. We seek to gain deeper insights into the estimation of extropy for the Weibull distribution where it is noted for its adaptability and application, it is commonly used to describe life-cycle data and dependability problems, see Baratpour *et al*. (2007), Chacko *et al*. (2021) and Murthy *et al*. (2004).

The pdf of Weibull distribution is defined respectively as follows:

$$f(x) = b\lambda x^{\lambda-1} e^{-bx^{\lambda}} \tag{3}$$

Understanding the extropy associated with Weibull-distributed records can enhance our ability to model and predict phenomena in various fields, from engineering to environmental studies.

2. Maximum likelihood estimation

The MLE method can be used to explore the range of possible distributions and parameters. it seeks to identify model parameter values that maximize the probability function over the parameter space. The maximum likelihood method is a widely used statistical inference technique that may be used to a variety of distributions and models. The Fisher information matrix (FIM) can be used to calculate confidence intervals (CIs) due to its asymptotic features.

1.1 Likelihood Equations

Let $D_n = U_1, \ldots, U_n$ be the first n upper record values from Weibull distribution $W(b, \lambda)$. from (1), the likelihood function, say, $L(b, \lambda; u)$, can be presented as

$$L(b, \lambda; u) = e^{-bu_n^{\lambda}} \prod_{i=1}^{n} b\lambda\, u_i^{\lambda-1} \tag{4}$$

The log-likelihood function is given by:

$$\begin{aligned} l(b, \lambda; u) &= -bu_n^{\lambda} + \sum_{i=1}^{n} \log b + \log\ \lambda + (\lambda - 1)\log u_i \\ &= -bu_n^{\lambda} + n\log b + n\log\ \lambda + \sum_{i=1}^{n} (\lambda - 1)\log u_i \end{aligned} \tag{5}$$

The partial derivatives of $l(b, \lambda; u)$ for b, λ is derived respectively as

$$\frac{\partial l(b,\lambda;u)}{\partial b} = -u_n^{\lambda} + \frac{n}{b} \tag{6}$$

$$\frac{\partial l(b,\lambda;u)}{\partial \lambda} = -b\lambda u_n^{\lambda-1} + \frac{n}{\lambda} + \sum_{i=1}^{n} \log u_i \tag{7}$$

Now, to get the MLEs of b and λ we set the equations (6) and (7) to zero. Therefore, the MLE of b is given by

$$\hat{b}_{ML} = \frac{n}{u_n^{\hat{\lambda}}} \tag{8}$$

For $\hat{\lambda}_{ML}$ cannot be derived with an explicit form. Therefore, we need to solve the nonlinear equation (7) numerically. One of the most used methods is the Newton–Raphson (N–R) method.

Using the invariant property, the MLE of J_X is :

$$J_X = -\hat{b}_{ML}^{\;-2\hat{\lambda}_{ML}+3}\,\frac{\hat{\lambda}_{ML}^{\;2}}{2^{\hat{\lambda}_{ML}^{\;2}}}\Gamma(2\hat{\lambda}_{ML}-1) \tag{9}$$

1.2 Asymptotic Confidence Intervals for MLEs

For more accuracy of MLEs, we use the asymptotic variance of MLE to determine the ACIs of b and λ. Let $I(\Phi)$ be the fisher information matrix where $\Phi = (b, \lambda)$, the FIM can be given as follows

$$I(\Phi) = \begin{bmatrix} -\frac{\partial^2 l(b,\lambda;u)}{\partial b^2} & -\frac{\partial^2 l(b,\lambda;u)}{\partial b \partial \lambda} \\ -\frac{\partial^2 l(b,\lambda;u)}{\partial b \partial \lambda} & -\frac{\partial^2 l(b,\lambda;u)}{\partial \lambda^2} \end{bmatrix} \tag{10}$$

Thus,

$$I(\Phi) = \begin{bmatrix} \frac{n}{b^2} & \lambda u_n^{\lambda-1} \\ \lambda u_n^{\lambda-1} & b u_n^{\lambda-1} + b\lambda(\lambda-1)u_n^{\lambda-2} + \frac{n}{\lambda^2} \end{bmatrix} \tag{11}$$

To find the $Var(\hat{b}_{ML})$ and $Var(\hat{\lambda}_{ML})$, we should calculate the inverse of FIM of the MLEs under the asymptotic property. Thus:

$$I(\hat{\Phi}) = \begin{bmatrix} \frac{n}{\hat{b}_{ML}^{\;2}} & \hat{\lambda}_{ML} u_n^{\hat{\lambda}_{ML}-1} \\ \hat{\lambda}_{ML} u_n^{\hat{\lambda}_{ML}-1} & \hat{b}_{ML} u_n^{\hat{\lambda}_{ML}-1} + \hat{b}_{ML}\hat{\lambda}_{ML}(\hat{\lambda}_{ML}-1)u_n^{\hat{\lambda}_{ML}-2} + \frac{n}{\hat{\lambda}_{ML}^{\;2}} \end{bmatrix} \tag{12}$$

Where, $\hat{\Phi}$ is the estimate of Φ. Thus

$$I(\hat{\Phi})^{-1} = \frac{1}{\det\left(I(\hat{\Phi})\right)} \begin{bmatrix} \hat{b}_{ML} u_n^{\hat{\lambda}_{ML}-1} + \hat{b}_{ML}\hat{\lambda}_{ML}(\hat{\lambda}_{ML}-1)u_n^{\hat{\lambda}_{ML}-2} + \frac{n}{\hat{\lambda}_{ML}^{\;2}} & -\hat{\lambda}_{ML} u_n^{\hat{\lambda}_{ML}-1} \\ -\hat{\lambda}_{ML} u_n^{\hat{\lambda}_{ML}-1} & \frac{n}{\hat{b}_{ML}^{\;2}} \end{bmatrix}$$
$$= \begin{bmatrix} Var(\hat{b}_{ML}) & COV(\hat{b}_{ML}, \hat{\lambda}_{ML}) \\ COV(\hat{b}_{ML}, \hat{\lambda}_{ML}) & Var(\hat{\lambda}_{ML}) \end{bmatrix} \tag{13}$$

Where

$$\det\left(I(\hat{\Phi})\right) = \frac{n}{\hat{b}_{ML}}\left(u_n^{\hat{\lambda}_{ML}-1} + \hat{\lambda}_{ML}(\hat{\lambda}_{ML}-1)u_n^{\hat{\lambda}_{ML}-2}\right) + \left(\frac{n}{\hat{b}_{ML}\hat{\lambda}_{ML}}\right)^2 - \left(\hat{\lambda}_{ML} u_n^{\hat{\lambda}_{ML}-1}\right)^2 \tag{14}$$

The $(1-\varepsilon)100\%$ confidence intervals for $\hat{b}_{ML}$, $\hat{\lambda}_{ML}$ are given as

$$\hat{b}_{ML} \pm Z_{\frac{\varepsilon}{2}}\sqrt{Var(\hat{b}_{ML})}\ ,\ \hat{\lambda}_{ML} \pm Z_{\frac{\varepsilon}{2}}\sqrt{Var(\hat{\lambda}_{ML})} \tag{15}$$

respectively, where $Z_{\frac{\varepsilon}{2}}$ is $Z_{\frac{\varepsilon}{2}}100\%$ the lower percentile of standard normal distribution.

$$J_X = -\hat{b}_{ML}^{\;-2\hat{\lambda}_{ML}+3}\,\frac{\hat{\lambda}_{ML}^{\;2}}{2^{\hat{\lambda}_{ML}^{\;2}}}\Gamma(2\hat{\lambda}_{ML}-1) \tag{16}$$

To attain the $100(1-Z_{\varepsilon})\%$ two-sided asymptotic approximation CIs for J_X. The delta method can be used to approximate the variances of extropy. Let:

$$D_{J_X} = \left(\frac{\partial J_X}{\partial b} \quad \frac{\partial J_X}{\partial \lambda}\right)_{b=\hat{b}_{ML},\ \lambda=\hat{\lambda}_{ML}} \tag{17}$$

Then we find the estimated variance of extropy as follows

$$Var(\widehat{J_X}) = D_{J_X}\, I(\widehat{\Phi})^{-1} {D_{J_X}}^{T} \tag{18}$$

Therefore

$$\left[\widehat{J_X} - Z_{\frac{\varepsilon}{2}}\sqrt{Var(\widehat{J_X})}\,, \widehat{J_X} + Z_{\frac{\varepsilon}{2}}\sqrt{Var(\widehat{J_X})}\right] \tag{19}$$

3. Bayes inference

In this section, we concentrate on the main objective which is the Bayesian estimation to estimate the parameters b and λ and also J_X. For this method we use the squared error and LINEX loss functions, it can be defined respectively as follows:

$$L_1(\Phi,\widehat{\Phi}) = (\Phi-\widehat{\Phi})^2,\ L_2(\Phi,\widehat{\Phi}) = \mathrm{e}^{(\eta(\Phi-\widehat{\Phi}))} - \eta(\Phi-\widehat{\Phi}) - 1 \tag{20}$$

Let's choose the prior distribution of b and λ. We propose the parameters independently follow a Gamma distribution ($b\sim Gamma(\alpha,\beta)$ and $\lambda\sim Gamma(\gamma,\tau)$, where $\alpha,\beta,\gamma, and\ \tau$ are positive real constants). Thus, the joint prior distribution

$$P(b,\lambda)\ \alpha\ \lambda^{\gamma-1}b^{\alpha-1}e^{-\lambda\tau-b\beta};\ \alpha,\beta,\gamma,\tau>0 \tag{21}$$

Hence, The joint posterior distribution

$$\begin{aligned} P^*(b,\lambda|u) &= \frac{L(b,\lambda;u)P(b,\lambda)}{\iint L(b,\lambda;u)P(b,\lambda)\,db\,d\lambda} \\ &= \frac{\lambda^{\gamma-1}b^{\alpha-1}e^{-\lambda\tau-b\beta-bu_n^{\lambda}}\prod_{i=1}^{n} b\lambda\, u_i^{\lambda-1}}{\iint e^{-bu_n^{\lambda}}\prod_{i=1}^{n} b\lambda\, u_i^{\lambda-1}\lambda^{\gamma-1}b^{\alpha-1}e^{-\lambda\tau-b\beta}\,db\,d\lambda} \end{aligned} \tag{22}$$

The joint posterior density can be written as:

$$P^*(b,\lambda|u)\ \alpha\ \lambda^{2n+\gamma-1}b^{2n+\alpha-1}\prod_{i=1}^{n} u_i^{\lambda-1}\ e^{-\lambda(\tau-u_n^{b})-b(\beta-\sum_{i=1}^{n}\log u_i-u_n^{\lambda})} \tag{23}$$

2.1 Markov Chain Monte Carlo

The Bayes estimates for determining the posterior mean for the parameters are difficult to get unless numerical approximation methods are used. There are numerous approximation approaches in the literature for dealing with this type of situation. We consider the (MCMC) approximation approach and the Gibbs sampling algorithm which are popular Bayesian estimating techniques that rely on marginal posterior distributions for sampling. Readers may refer to Pradhan *et al*. (2011), Chib *et al*. (1995) and Al-Labadi *et al*. (2020).

The full conditional posterior distributions for b and λ s are as follows:

$$P^{*}(\lambda|b,u)\ \alpha\ \lambda^{2n+\gamma-1}\prod_{i=1}^{n}u_i^{\lambda-1}\ e^{-\lambda(\tau-u_n^{b})+bu_n^{\lambda}} \tag{24}$$

$$P^{*}(b|\lambda,u)\ \alpha\ b^{2n+\alpha-1}e^{\lambda u_n^{b}-b(\beta-\sum_{i=1}^{n}\log u_i-u_n^{\lambda})} \tag{25}$$

Thus, we must use the Metropolis–Hastings (M–H) algorithm to generate the unknown parameters because the densities in Equations (24,25) cannot be written as known densities. As a result, it is impossible to generate b and λ directly from these densities using conventional methods; for more information. The M–H algorithm aims to minimize rejection rates as much as possible. To find the (BEs) and generate credible intervals for the required parameters, the M–H algorithm relies on selecting the normal distribution. The Gibbs technique, which can be summarized as the following algorithm, is as follows:

Step 1: Put the ML estimators of b and λ as initial values b_0 and λ_0.

Step 2: *Let* $T=1,\dots.,N$ be the observations generated from the conditional posterior distributions for b and λ (24) and (25) respectively.

Step 3: Repeat Steps 2 M times to obtain MCMC samples $(b^1,\lambda^1),\dots,(b^M,\lambda^M)$ **where** M The total amount of cycles needed.)**.**

Step 4: The Bayes estimator of extropy given in (9) under SE and LINEX are presented as follow

$$\widehat{J_{SE}}=\frac{1}{M-m}\sum_{t=m+1}^{M}-b^{t^{-2\lambda^t+3}}\frac{(\lambda^t)^2}{2^{(\lambda^t)^2}}\Gamma(2\lambda^t-1) \tag{26}$$

$$\widehat{J_{LX}}=\frac{-1}{\eta}\ log\left[\frac{1}{M-m}\sum_{t=m+1}^{M}e^{\eta b^{t^{-2\lambda^t+3}}\frac{(\lambda^t)^2}{2^{(\lambda^t)^2}}\Gamma(2\lambda^t-1)}\right] \tag{27}$$

where m is the first iterations as burn in period.

4. Simulation

A simulation study was carried out to assess the performance of the estimating techniques created in Sections 2 and 3. First, we get the extropy of k-records for unknown parameters $b,\ \lambda$ at equation (9) by calculating the MLE $\hat{b}_{ML}$ and $\hat{\lambda}_{ML}$ which we were able to obtain the bias of MLEs for various values k . Now, assuming the model parameters $\alpha=2,\ \beta=2$ and $\gamma=2,\tau=2,$ 500 observations. Based on these data sets, the maximum likelihood estimates (MLEs) and Bayes estimates of the parameters were obtained. For Bayesian estimation, we generated 10,000 realizations of the Markov chains using the Gibbs and Metropolis–Hastings algorithms. The results of the simulation study are summarized in Table 1. We observe that the bias of all estimators decrease as the sample size n increases and the bias of bayes estimation under the SE loss function is smaller than the MLE.

Table 1 – The bias of MLE and BE for Weibull distribution

k	Shape	Scale	Extropy	MLE	Bayes	
					SEL	LX: h=1
				|Bias|	|Bias|	|Bias|
6	1.5	2	0.078998995	0.006430693	0.006030694	0.012340693
7	1.5	2	0.072187881	0.013241806	0.012141809	0.019671806
8	1.5	2	0.077499109	0.007930578	0.007820571	0.009830578
9	1.5	2	0.067900459	0.017529228	0.017419221	0.018929228
6	1.5	2.5	0.078998995	0.047277892	0.04417789	0.051277892
7	1.5	2.5	0.072187881	0.054089006	0.051289009	0.062289006
8	1.5	2.5	0.077499109	0.048777778	0.042377772	0.057877778
9	1.5	2.5	0.06790046	0.058376427	0.054276421	0.061276427
6	1.6	2	0.118035454	0.016182274	0.011682279	0.019682274
7	1.6	2	0.106579012	0.027638716	0.022238717	0.027638716
8	1.6	2	0.115357326	0.018860402	0.013160403	0.018860402
9	1.6	2	0.09954352	0.034674208	0.031674201	0.034674208
6	1.6	2.5	0.118035454	0.093583058	0.091783051	0.093583058
7	1.6	2.5	0.106579012	0.1050395	0.102234394	0.196620395
8	1.6	2.5	0.115357325	0.096261187	0.091161189	0.126261187
9	1.6	2.5	0.09954352	0.112074992	0.100074993	0.187874992

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