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Vibration analysis using the theory of exponential shear deformation for laminated plates

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Abstract
In this research, the free vibration response of laminated composite plates is studied using a refined exponential shear strain theory. The most interesting feature of this theory is that it allows exponential distributions of transverse shear strains, and verifies zero-shear boundary conditions on the plate surfaces without the use of shear correction factors. Some stiffness constants are written as a series. The number of independent unknowns in the present theory is four, compared with five in other shear deformation theories. The equations of motion are obtained from Hamilton's principle and Navier's method is used to determine the exact solution for antisymmetric cross-laminated plates. The numerical results found in the present analysis for free vibration are presented and compared with those available in the literature. The proposed theory is not only accurate, but also effective in predicting the fundamental frequencies of laminated composite plates.
Keywords: Laminated composite plates. Free vibration. Refined exponential theory. Navier solution.

1. Introduction

Today, composite materials are used in almost every phase of structural work, from spacecraft to ships, from bridges to domes on civic buildings. The significant increase in the use of composite structures requires the development of rigorous mathematical methods capable of modelling, designing and optimizing the composite under any set of conditions (Karama *et al.*, 2003; Xiao *et al.*, 2024; Zhuangzhuang *et al.*, 2023). Due to the high degrees of anisotropy and low stiffness in plate transverse shear, the Kirchhoff hypothesis as a classical theory is no longer adequate. The hypothesis states that the normal to the median plane of a plate remains straight and normal after deformation due to the negligible effects of transverse shear. The free vibration frequencies calculated using classical thin plate theory are higher than those obtained using Mindlin plate theory, in which the effects of transverse shear and rotational inertia are included (Nedri *et al.*, 2014; Aniket *et al.*, 2022).

Classical plate theory (CLPT), which does not introduce transverse shear deformation effects, gives reasonable results for thin plates. In order to overcome the problem encountered in CLPT, shear deformation theories that take into account the effect of transverse shear deformation have been proposed. First-order shear deformation theory (FSDT) considers the linear distribution of transverse shear stresses and strains across the thickness. Much research into the free vibrations of laminated composite sheets has been carried out using first-order shear strain theory (FSDT) (Atteshamuddin *et al.*, 2015; Draiche *et al.*, 2019; Khdeir, 1989; Noor *et al.*, 1989; Whitney *et al.*, 1970). To overcome the limitations of classical plate theory (CPT) and first-order shear deformation theory (FSDT), high-order shear deformation theory (HSDT) was recommended. High-order shear deformation theory (HSDT) has been widely used to study the vibration behaviour of composite structures.

Shimpi (2002) has developed a refined model for isotropic plates (RPT: Refined Plate Theory). The most interesting feature of this method is that it contains only two variables with only four unknowns to find instead of the minimum five unknowns for other theories. In addition, this theory does not require a shear correction factor and gives a parabolic distribution of transverse shear stresses and strains through the plate thickness. Recently, Thai *et al.* (2010), Thai *et al.* (2011) and Hadji *et al.* (2011) have successfully adapted the Refined Plate Theory (RPT) to the buckling of orthotropic plates and free vibrations of laminated and sandwich plates respectively (Bachir, Bouiadjra, 2015).

2. Mathematical analysis of the problem

Consider a laminated composite sheet of total thickness h, length a and width b. This is made up of n orthotropic elastic layers. The x and y coordinates are in the plane (O_x, O_y), and the axis O_z is along the thickness. The top and bottom surfaces of the plate are at $z = \pm h/2$ and the edges of the plate are parallel to the axes O_x and O_y; as shown in Figure 1.

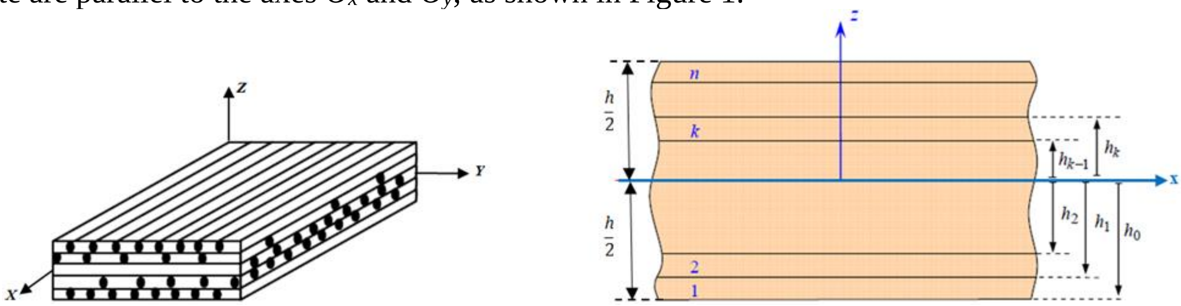


Figure 1 - Representation of the laminated composite plate and the coordinate systems.

2.1 Refined theory of exponential shear deformation of laminated composite plates

Unlike other shear deformation theories, the number of unknowns in the present theory is only four. In addition, this theory does not require a shear correction coefficient and satisfies the condition of zero transverse shear stresses at the plate edges.

$$\begin{cases} u(x,y,z,t) = u_0(x,y,t) - z\dfrac{\partial}{\partial x}w_b(x,y,t) - f(z)\dfrac{\partial}{\partial x}w_s(x,y,t) \\ v(x,y,z,t) = v_0(x,y,t) - z\dfrac{\partial}{\partial y}w_b(x,y,t) - f(z)\dfrac{\partial}{\partial y}w_s(x,y,t) \\ w(x,y,z,t) = w_b(x,y,t) + w_s(x,y,t) \end{cases} \tag{1}$$

$$f(z) = z - ze^{-2\left(\frac{z}{h}\right)^2} \tag{2}$$

$f(z)$: is the warp function representing the variation in transverse shear stresses and strains across the thickness.

2.2 Formulation of the equations of motion

Hamilton's principle is used to determine the equations of motion. The principle can be stated in analytical form (Reddy, 2002; Draiche *et al.*, 2014), and the equation of motion for a laminated composite plate is written as above:

$$\int_{t_1}^{t_2} (\delta\, \mathrm{U} - \delta\, \mathrm{K})\, dt = 0 \tag{3}$$

The variation in the deformation energy of the laminated composite plate can be written as follows:

$$\begin{aligned}\delta\, \mathrm{U} = \iint \Big[& N_x \delta\, \varepsilon_x^0 + N_y \delta\, \varepsilon_y^0 + N_{xy} \delta\, \gamma_{xy}^0 + M_x^b \delta\, k_x^b + M_y^b \delta\, k_y^b + M_{xy}^b \delta\, k_{xy}^b \\ & + M_x^s \delta\, k_x^s + M_y^s \delta\, k_y^s + M_{xy}^s \delta\, k_{xy}^s + S_{yz}^s \delta\, \gamma_{yz}^s + S_{xz}^s \delta\, \gamma_{xz}^S \Big] dxdy\end{aligned} \tag{4}$$

The variation in the kinetic energy of the laminated composite plate can be calculated as follows:

$$\begin{aligned}\delta\, \mathrm{K} = \iint \Big\{ & I_0 [\dot{u}_0 \delta \dot{u}_0 + \dot{v}_0 \delta \dot{v}_0 + (\dot{w}_b + \dot{w}_s) \delta (\dot{w}_b + \dot{w}_s)] - I_1 \left(\dot{u}_0 \frac{\partial \delta \dot{w}_b}{\partial x} + \frac{\partial \dot{w}_b}{\partial x} \delta\, \dot{u}_0 + \dot{v}_0 \frac{\partial \delta \dot{w}_b}{\partial y} + \frac{\partial \dot{w}_b}{\partial y} \delta\, \dot{v}_0 \right) \\ & + I_2 \left(\frac{\partial \dot{w}_b}{\partial x} \frac{\partial \delta\, \dot{w}_b}{\partial x} + \frac{\partial \dot{w}_b}{\partial y} \frac{\partial \delta\, \dot{w}_b}{\partial y} \right) - I_3 \left(\dot{u}_0 \frac{\partial \delta \dot{w}_s}{\partial x} + \frac{\partial \dot{w}_s}{\partial x} \delta\, \dot{u}_0 + \dot{v}_0 \frac{\partial \delta \dot{w}_s}{\partial y} + \frac{\partial \dot{w}_s}{\partial y} \delta\, \dot{v}_0 \right) \\ & + I_4 \left(\frac{\partial \dot{w}_s}{\partial x} \frac{\partial \delta\, \dot{w}_s}{\partial x} + \frac{\partial \dot{w}_s}{\partial y} \frac{\partial \delta\, \dot{w}_s}{\partial y} \right) + I_5 \left(\frac{\partial \dot{w}_b}{\partial x} \frac{\partial \delta\, \dot{w}_s}{\partial x} + \frac{\partial \dot{w}_s}{\partial x} \frac{\partial \delta \dot{w}_b}{\partial x} + \frac{\partial \dot{w}_b}{\partial y} \frac{\partial \delta\, \dot{w}_s}{\partial y} + \frac{\partial \dot{w}_s}{\partial y} \frac{\partial \delta \dot{w}_b}{\partial y} \right) \Big\} dxdy\end{aligned} \tag{5}$$

The equations of motion at dynamic equilibrium for a freely oscillating antisymmetric laminated composite plate are derived as follows:

$$\begin{aligned}
\delta\, u_0 &: \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \ddot{u}_0 \\
\delta\, v_0 &: \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0 \ddot{v}_0 \\
\delta\, w_b &: \frac{\partial^2 M_x^b}{\partial x^2} + 2 \frac{\partial^2 M_{xy}^b}{\partial x \partial y} + \frac{\partial^2 M_y^b}{\partial y^2} = I_0 \left(\ddot{w}_b + \ddot{w}_s \right) - I_2 \left(\frac{\partial^2 \ddot{w}_b}{\partial x^2} + \frac{\partial^2 \ddot{w}_b}{\partial y^2} \right) - I_5 \left(\frac{\partial^2 \ddot{w}_s}{\partial x^2} + \frac{\partial^2 \ddot{w}_s}{\partial y^2} \right) \\
\delta\, w_s &: \frac{\partial^2 M_x^s}{\partial x^2} + 2 \frac{\partial^2 M_{xy}^s}{\partial x \partial y} + \frac{\partial^2 M_y^s}{\partial y^2} + \frac{\partial Q_{xz}^s}{\partial x} + \frac{\partial Q_{yz}^s}{\partial y} = I_0 \left(\ddot{w}_b + \ddot{w}_s \right) - I_4 \left(\frac{\partial^2 \ddot{w}_s}{\partial x^2} + \frac{\partial^2 \ddot{w}_s}{\partial y^2} \right) \\
& - I_5 \left(\frac{\partial^2 \ddot{w}_b}{\partial x^2} + \frac{\partial^2 \ddot{w}_b}{\partial y^2} \right)
\end{aligned} \tag{6}$$

3. Results

In this analysis; the aim is to demonstrate the accuracy of the present method developed for dynamic response and the correlation between this present theory containing a displacement field with only four unknowns and exponential distributions of stresses and transverse shear strains through the plate thickness in antisymmetric laminated composites and plate theories is established through case studies of two materials 1 and 2 which we will develop. The description of the different displacement models is presented in Table 1.

Table 1. Displacement model

Models	Theory	Unknown Variables
CLPT	Classical Plate Theory	3
FSDT	First-order shear deformation theory (Whitney *et al.*, 1970)	5
TSDT	Third-Order Shear Deformation Theory (Reddy, 1997)	5
HSDT	Higher-Order Shear Deformation Theory (Swaminathan *et al.*, 2008)	12
ESDT	Exponential Shear Deformation Theory (Karama *et al.*,2003)	5
Présente	Refined Exponential Shear Deformation Theory	4

The properties of the materials used are given in Table 2.

Table 2. Characteristics of orthotropic materials.

Material 1 (Noor, 1973)			
E_1/E_2	$G_{12}=G_{13}$	G_{23}	ν_{12}
variable	0.6 E_2	0.5 E_2	0.25

For first order shear deformation theory (FSDT) the shear correction coefficient of 5/6 is considered. All layers have the same thickness, mass density and orthotropic material characteristics along the principal axes of the materials. The vibrational responses of the antisymetrical laminated plates allow us to determine the dimensionless frequencies.

3.1. Effects of orthotropy ratio and degree of orthotropy on dimensionless frequencies for a length-to-thickness ratio a/h = 5.

Figure 2 shows the influence of the orthotropy ratio E_1 / E_2 on the dimensionless frequencies $\bar{\omega}$ for square plates made of two-layer $(0^\circ/90^\circ)_1$ and ten-layer $(0^\circ/90^\circ)_5$ antisymmetric cross-laminated composites with $a/h=5$.

This result shows that as the degree of orthotropy *n* increases, the dimensionless frequency $\bar{\omega}$ will remain the same.

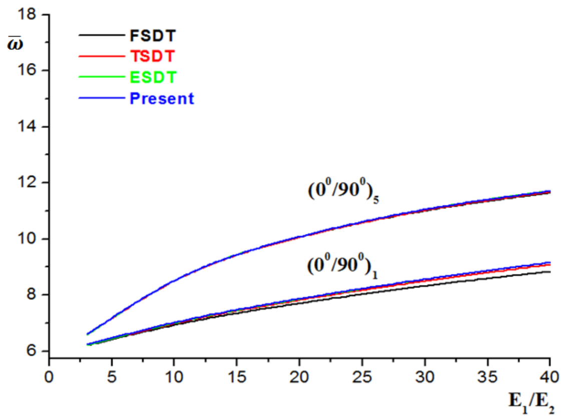


Figure 2 - Effects of orthotropy ratio and degree of orthotropy on the dimensionless frequencies of a square plate of antisymmetric cross-laminated composites with a/h = 5.

3.2. Effects of the length-to-thickness ratio and the degree of orthotropy on the dimensionless frequencies for an orthotropy ratio E_1/E_2 = 40.

Figure 3 shows the effects of the length-to-thickness ratio a/h and the degree of orthotropy n on the dimensionless frequencies $\overline{\omega}$ for square plates of two-layer and six-layer antisymmetric cross-laminated composites with E_1/E_2 = 40. This result shows that when the ratio of length to thickness a/h increases, the dimensionless frequency increases and the values obtained are the same for all the theories, confirming that we are dealing with thin plates. As the degree of orthotropy increases, the values obtained are identical except for the values obtained by the classical theory, which does not take into account the effect of transverse shear deformation.

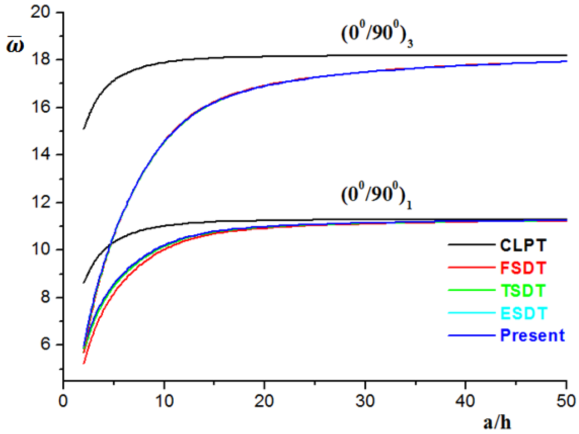
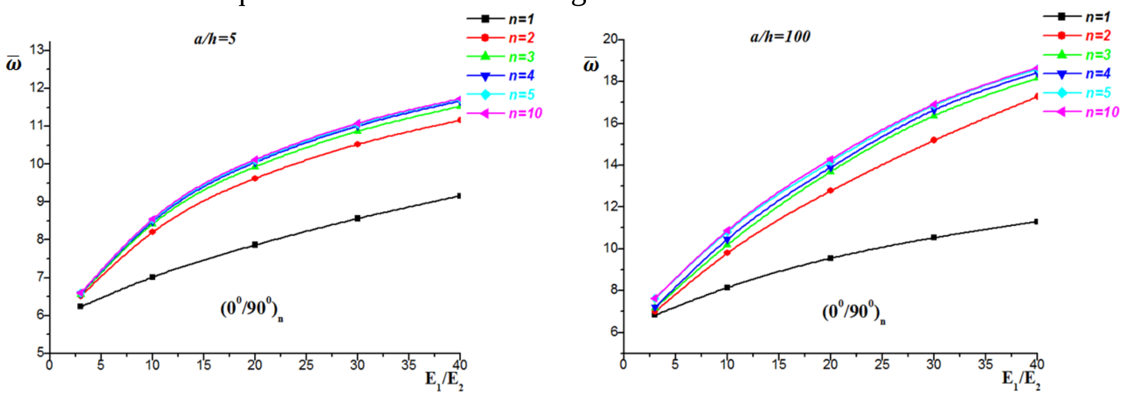


Figure 3 - Effects of length-to-thickness ratio and degree of orthotropy on the dimensionless frequencies of a square plate of antisymmetric cross-laminated composites with E_1/E_2 = 40.

For length-to-thickness ratios of less than 10, the effect of transverse shear deformation is very significant, demonstrating that we are dealing with thick plates. Above a value of *a/h*=10 and for high degrees of orthotropy, the values of the dimensionless frequency are almost constant and identical for all the theories except for the classical theory, which does not take into account the effect of transverse shear deformation. As the degree of orthotropy increases, the difference between the dimensionless frequency values obtained by the theories (RESDT, FSDT, TSDT and ESDT) decreases significantly, except for the classical theory (CLPT).

3.3. Effects of orthotropy ratio and degree of orthotropy on dimensionless frequencies for two types of thick and thin plates

The influence of the orthotropic ratio E_1/E_2 and the degree of orthotropy n on the dimensionless frequencies $\overline{\omega}$ is shown in Figure 4.



(a) *(b)*

Figure 4 - Effects of orthotropy ratio and degree of orthotropy on the dimensionless frequencies of two square plates of antisymmetric cross-laminated composites with (a): *a/h*=5 and (b): *a/h*=100.

Analysis of the variation of the dimensionless frequency as a function of the orthotropy ratio ***E_1/E_2*** for two types of thick and thin plates for the same values of the degree of orthotropy (1; 2; 3; 4; 5; 10) shows that increasing the degree of orthotropy produces an increase in the dimensionless frequency. It can be seen that the dimensional frequencies are much higher for thin plates than for thick plates, and this is due to the transverse shear effect, which is more significant for thick plates.

3.4. Effects of length-to-thickness ratio and degree of orthotropy on dimensionless frequencies for two types of weakly orthotropic and strongly orthotropic plates

The effects of the length-to-thickness ratio *a/h*, and the degree of orthotropy *n* on the free vibrations of laminated composite sheets is illustrated in Figure 5.

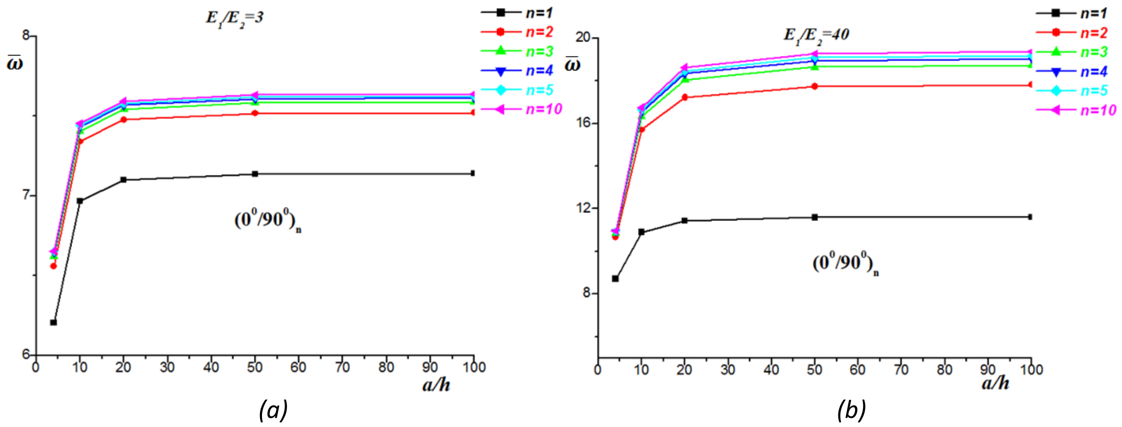


Figure 5 - Effects of length-to-thickness ratio and degree of orthotropy on the dimensionless frequencies of two square plates of antisymmetric cross-laminated composites with (a): *E_1/E_2*=3 and (b): *E_1/E_2*=40.

Figure 5 shows that the dimensionless frequency of free vibrations increases as the length-to-thickness ratio and the degree of orthotropy increase. As the orthotropy ratio increases, the dimensionless frequency increases, and it is all the greater for highly orthotropic plates.

5. Conclusions

- In this analysis, we have developed a refined exponential shear deformation theory for the free vibrations of simply supported plates in antisymmetric laminated composites,
- This theory takes into account an exponential stress distribution across the thickness and checks the boundary conditions (top and bottom surfaces of the plate) without the use of a shear correction factor,
- In this theory, the warping function has been developed in the form of a non-polynomial series that does not use Taylor's limited expansion,
- The dimensionless frequencies are strongly influenced by the variation in the orthotropy ratio, the degree of orthotropy and the thickness of the plate (ratio of length to thickness),
- The effect of shear has a strong influence on the rigidity of the plates.

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