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A Dynamic Electro-Elastic Viscoplastic Contact Problem with an Internal Variable

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Ahmed Hamidat
ORCID: https://orcid.org/0000000276376413
Laboratory of Operator Theory and PDE: Foundations and Applications, Faculty of Exact Sciences, University of El Oued, 39000, El Oued, Algeria
E-mail: hamidat-ahmed@univ-eloued.dz
Hakim Bagua
ORCID: https://orcid.org/0000000251658609
Department of Electronics and Telecommunication, Faculty of New Technologies of Information and Communication, University of Ouargla, 30000, Ouargla, Algeria
E-mail: bagua.hakim@univ-ouargla.dz

Abstract
In this paper, we examine a dynamic contact problem involving an electro-elastic-viscoplastic body and a deformable base. The contact is characterized by an instantaneous normal response, and the behavior is described by an electro-elastic-viscoplastic law with an internal variable. We present both the mechanical and variational formulations of the problem, establishing the existence and uniqueness of the solution. Our proof relies on the theory of variational equations and fixed-point arguments.
Keywords: Electro-viscoplastic. Dynamic. Differential equations. Fixed point.

1. Introduction

Contact mechanics is a comprehensive field that encompasses various phenomena involving the contact between a deformable or solid body and a foundation. Our contributions to this field are detailed in references [5, 6, 7, 8, 13]. These contact phenomena are prevalent in everyday life, significantly impacting structural mechanics, particularly in industries such as automotive and aeronautics (e.g., cracks in composites and fiber/matrix interfaces), energy production (e.g., assembly of structures, welding joint failures), and transmission systems. As a result, substantial efforts have been devoted to their modeling and numerical simulation, as demonstrated by works such as [4, 11, 12, 14] and related literature.

Piezoelectricity is a property of certain materials that allows them to become electrically polarized when subjected to mechanical stress, known as the direct piezoelectric effect. Conversely, the inverse piezoelectric effect occurs when these materials deform in response to an electric field.

In practical applications, the direct effect enables the creation of sensors, such as pressure sensors. The inverse effect is vital for producing actuators, including piezoelectrically controlled injectors used in automobiles, piezoelectric dampers essential for reducing vibrations, and ultrasonic transducers widely utilized in medical imaging, particularly in ultrasound applications.

Many crystalline materials exhibit piezoelectric behavior, with some showing sufficiently strong effects for various applications. Examples include quartz, Rochelle salt, lead titanate zirconate ceramics, barium titanate, and polyvinylidene fluoride (a polymer film). These materials are classified as intelligent materials due to their adaptive nature, scalability, and the combination of material properties with functional applications.

The constitutive laws with internal variables have been employed in various publications to model the effect of these variables on the behavior of real bodies such as metals, rocks, polymers, and others, where the rate of deformation depends on the internal variables. Some internal state variables considered by many authors include the spatial distribution of dislocations, material work-hardening, absolute temperature, and the damage field. For examples and detailed discussions on hardening, temperature, and other internal state variables, refer to [9, 3] and the references therein. This article contributes to the analysis of contact problem between deformable bodie and foundations, examining dynamic processes for materials such as electro-elastic viscoplastics with internal variables. The boundary conditions involve an instantaneous normal response, and the friction laws used are versions of Coulomb's law. Electrical conditions are introduced for insulating foundations. Our study of contact phenomena includes mathematical modeling and variational analysis, with results on the existence and uniqueness of solutions.

In Section 2, we introduce the contact model and provide insights on the associated boundary conditions. Section 3 outlines the assumptions regarding the data and formulates the variational description. Section 4 is dedicated to proving the existence and uniqueness of the solution.

2. Problem statement

The physical setting is the following. Let us consider electro- thermo-elastic-viscoplastic bodie occupies a bounded domain $\Omega \subset \mathbb{R}^d (d = 2,3)$ with a smooth boundary Γ, Let $T > 0$ and let $[0,T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0,T)$, so the displacement field vanishes there, surface tractions of density f_0 act on $\Gamma_2 \times (0,T)$ and a volume force of density f_2 is applied in $\Omega \times (0,T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0,T)$ and a surface electrical charge of density q_2 is prescribed on $\Gamma_b \times (0,T)$.

The classical formulation of the mechanical problem of electro-elastic-viscoplastic material with internal state variable, may be stated as follows.

Problem $\boldsymbol{P}$

Find a displacement field $\boldsymbol{u}: \Omega \times (0,T) \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}: \Omega \times (0,T) \to \mathbb{S}^d$, an electric potential field $\varphi: \Omega \times (0,T) \to \mathbb{R}$, an electric displacement field $\boldsymbol{D}: \Omega \times (0,T) \to \mathbb{R}^d$, and an internal state variable field $\boldsymbol{k}: \Omega \times (0,T) \to \mathbb{R}^m$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{B}\big(\varepsilon(\boldsymbol{u}(t))\big) - (\mathcal{E})^* E(\varphi(t)) + \int_0^t \mathcal{G}(\sigma(s) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s)) + (\mathcal{E})^* E(\varphi(t)), \varepsilon(\boldsymbol{u}(s)), \boldsymbol{k}(s))ds \text{ in } \Omega \times (0,T), \quad (1)$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{B}\nabla(\varphi) \quad \text{in } \Omega \times (0,T), \quad (2)$$

$$\dot{\boldsymbol{k}} = \Theta(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s)) + (\mathcal{E})^* E(\varphi(t)), \varepsilon(\boldsymbol{u}), \boldsymbol{k}) \quad \text{in } \Omega \times (0,T), \quad (3)$$

$$div\, \boldsymbol{\sigma} + f_0 = \rho \ddot{u} \quad \text{in } \Omega \times (0,T), \quad (4)$$

$$div\, \boldsymbol{D} - q_0 = 0 \quad \text{in } \Omega \times (0,T), \quad (5)$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0,T), \quad (6)$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = f_2 \qquad \text{on } \Gamma_2 \times (0,T), \tag{7}$$

$$\begin{cases} -\sigma_\nu = p_\nu(\dot{u}_\nu) \quad \text{on } \Gamma_3 \times (0,T) & (8) \\ \boldsymbol{\sigma}_\tau = 0 & (9) \end{cases}$$

$$\varphi = 0 \qquad \text{on } \Gamma_a \times (0,T), \tag{10}$$

$$\boldsymbol{D}.\nu = q_2 \qquad \text{on } \Gamma_b \times (0,T), \tag{11}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \boldsymbol{k}(0) = \boldsymbol{k}_0, \qquad \text{in } \Omega. \tag{12}$$

First, equations (1)-(3) represent electro elastic-viscoplastic constitutive law with internal state variable, were $\mathcal{A}$ is the viscosity operator, allowed to be nonlinear, $\mathcal{B}$ is the elasticity operator and $\mathcal{G}$ is a nonlinear constitutive function describing the viscoplastic behavior of the material and depending on the internal state variable $\boldsymbol{k}$, and Θ is a nonlinear function also depending on the internal state variable $\boldsymbol{k}$. The set of admissible internal state variables is defined by:

$$Y = \{a = (a_i): a_i \in L^2(\Omega), 1 \leq i \leq m\}.$$

$E(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represent the third order piesoelectric tensor, $\mathcal{E}^*$ is its transposition. Equations (4) and (5) represent the equilibrium equations for the stress and electric displacement fields, where $\rho: \Omega \to \mathbb{R}_+$ designates the mass density. The evolutionary processes defined by (4) are called dynamic processes. In some situations, this equation can be further simplified. For example, in the case where $\dot{u} = 0$, it is a static process. In the case where the velocity field varies slowly with respect to time, i.e. the term $\rho\ddot{u}$ can be neglected, we are in the presence of a quasistatic process. In these two cases the equation (4) becomes $Div\sigma + f_0 = 0 \quad \text{in} \quad \Omega \times (0,T)$. Equations (6)-(7) are the displacement-traction conditions. The relations (8)-(9) represent the contact conditions with instantaneous normal response, Where p_ν is a function given below. (10) and (11) represent the electric boundary conditions. Finally, (12) is the initial condition.

3. Variational formulation and preliminaries

Here are some notations and conventions that will be used throughout this paper. We denote the space of symmetric tensors of order two on $\mathbb{R}^d$ by $\mathbb{S}^d$ (where $d = 2,3$); $(,)$ and $\|.\|$ represent the scalar product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. Thus, we have:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d \quad \text{and} \quad \boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d,$$
$$\| \boldsymbol{u} \| = (\boldsymbol{u} \cdot \boldsymbol{u})^{\frac{1}{2}}, \quad \forall\, \boldsymbol{u} \in \mathbb{R}^d \quad \text{and} \quad \| \boldsymbol{\sigma} \| = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{\frac{1}{2}}, \quad \forall \boldsymbol{\sigma} \in \mathbb{S}^d.$$

In this context and henceforth, the indices i, j, and k will range from 1 to d, and the summation convention will apply to repeated indices. We denote the normal and tangential components of $\boldsymbol{v}$ on the boundary by v_ν and $\boldsymbol{v}_\tau$, given by

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu},$$

$$\boldsymbol{v}_\tau = \boldsymbol{v} - u_\nu \boldsymbol{\nu}.$$

We denote the stress field by $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{x}; t)$, the displacement field by $\boldsymbol{u} = \boldsymbol{u}(x; t)$, and the field of infinitesimal deformations by $\varepsilon(\boldsymbol{u})$. To simplify the notation, we will not explicitly indicate the dependence of these functions on $x \in \overline{\Omega}$ and $t \in [0, T]$.

For a stress field $\boldsymbol{\sigma}$, we denote the normal and tangential components at the boundary by $\boldsymbol{\sigma}_\nu$ and $\boldsymbol{\sigma}_\tau$, defined as follows:

$$\sigma_\nu = \sigma\nu \cdot \boldsymbol{\nu},$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}.$$

Let’s consider the Hilbert space

$$H^1(\Omega) = \{u \in L^2(\Omega) | \partial_i u \in L^2(\Omega), i = 1, \ldots, d\}.$$

We define the following spaces:

$$H = L^2(\Omega; \mathbb{R}^d), \quad H_1 = \{\boldsymbol{u} \in H | \varepsilon(\boldsymbol{u}) \in \mathcal{H}\} = H^1(\Omega; \mathbb{R}^d),$$

$$\mathcal{H} = \left\{\boldsymbol{\sigma} = \left(\sigma_{ij}\right) | \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\} = L^2(\Omega; \mathbb{S}^d), \quad \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} | Div\, \boldsymbol{\sigma} \in H\}.$$

The spaces H, H_1; $\mathcal{H}$, and $\mathcal{H}_1$ are real Hilbert spaces endowed with the scalar products given by

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i dx \quad \forall \boldsymbol{u}, \boldsymbol{v} \in H, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx \quad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H},$$

$$(\mathbf{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in H_1,$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div\boldsymbol{\sigma}, Div\boldsymbol{\tau})_H, \quad \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1.$$

Here, $\varepsilon: H^1(\Omega; \mathbb{R}^d) \to L^2(\Omega; \mathbb{S}^d)$ and $Div: \mathcal{H}_1 \to L^2(\Omega; \mathbb{R}^d)$ are the deformation and divergence operators, respectively, defined as follows:

$$\varepsilon(\mathbf{u}) = \left(\varepsilon_{ij}(\mathbf{u})\right),$$

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right),$$

$$Div(\boldsymbol{\sigma}) = \sigma_{ij,j}.$$

The norms associated with the spaces H, H_1, $\mathcal{H}$, and $\mathcal{H}_1$ are denoted by $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$, and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Given that the boundary Γ is Lipschitz, the exterior normal vector ν is well-defined almost everywhere on the boundary. For any vector field $\mathbf{v} \in H_1$, we use $\mathbf{v}$ to represent the trace $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ. The trace map $\gamma: H_1 \to H_\Gamma$ is linear and continuous but not surjective. The image of H_1 under this map is denoted by H_Γ, which continuously injects into $L^2(\Gamma)^d$. Let H'_Γ be the dual space of H_Γ, and let $(\cdot,\cdot)$ represent the duality pairing between H'_Γ and H_Γ. For every $\boldsymbol{\sigma} \in \mathcal{H}_1$, there exists an element $\boldsymbol{\sigma} \cdot \nu \in H'_\Gamma$ such that

$$(\boldsymbol{\sigma} \cdot \nu, \gamma\mathbf{v}) = (\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (Div\, \boldsymbol{\sigma}, \mathbf{v})_H \quad \forall\, \mathbf{v} \in H_1.$$

If $\boldsymbol{\sigma}$ is sufficiently regular (e.g., C^1), then we have the following formula:

$$(\boldsymbol{\sigma} \cdot \nu, \gamma \mathbf{v}) = \int_\Gamma \boldsymbol{\sigma} \cdot \nu \cdot \mathbf{v}\, da \quad \forall \mathbf{v} \in H_1.$$

For sufficiently regular $\boldsymbol{\sigma}$, we obtain Green's formula:

$$(\boldsymbol{\sigma}, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (Div\boldsymbol{\sigma}, \mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma} \cdot \nu \cdot \mathbf{v}\, da \quad \forall\, \mathbf{v} \in H_1.$$

We define the closed subspaces of $L^2(\Omega)$ and H_1 as follows:

$$V = \{\boldsymbol{v} \in H_1 | \boldsymbol{v} = 0 \text{ on } \Gamma_1\}. \tag{13}$$

Since $\text{meas}(\Gamma_1) > 0$, Korn's inequality holds on V. Thus, there exists a constant $C_K > 0$, depending only on Ω and Γ_1, such that:

$$\| \varepsilon(\boldsymbol{v}) \|_{\mathcal{H}} \geq C_K \| \boldsymbol{v} \|_{H^1(\Omega)^d} \quad \forall\, \boldsymbol{v} \in V.$$

A proof of Korn's inequality can be found in [10], p.79.

We then consider the inner product and associated norm defined by:

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \tag{14}$$

$$\| \boldsymbol{v} \|_V = \| \varepsilon(\boldsymbol{v}) \|_{\mathcal{H}} \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{15}$$

Thus, the norms $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent on V, making $(V, \|\cdot\|_V)$ a real Hilbert space. Additionally, by applying the Sobolev trace theorem and equation (14), there exists a constant $c_0 > 0$, depending only on Ω, Γ_1, and Γ_3, such that:

$$\| \boldsymbol{v} \|_{L^2(\Gamma 3)^d} \leq c_0 \| \boldsymbol{v} \|_V \quad \forall\, \boldsymbol{v} \in V. \tag{16}$$

In what follows, we define the Sobolev spaces associated with the electrical unknowns (field of the electrical displacement **D** and the electrical potential φ) of the electro-mechanical problem which will be introduced in this paper. Let the spaces

$$\mathcal{W} = \{\mathbf{D} = (D_i) | D_i \in L^2(\Omega), div\, \mathbf{D} \in L^2(\Omega)\}, \tag{17}$$

$$W = \{\xi \in H^1(\Omega), \xi = 0 \text{ on } \Gamma_a\}. \tag{18}$$

where $div\mathbf{D} = (D_{i,i})$. These spaces $\mathcal{W}$ and W are real Hilbert spaces endowed with the scalar products given by

$$(\mathbf{D}, \mathbf{E})_{\mathcal{W}} = (\mathbf{D}, \mathbf{E})_H + (div\, \mathbf{D}, div\, \mathbf{E})_{L^2(\Omega)}, \tag{19}$$

$$(\varphi, \xi)_W = \int_\Omega \nabla\varphi . \nabla\xi dx, \tag{20}$$

and the associated norms $\|.\|_{\mathcal{W}}$ and $\|.\|_W$, respectively.

$$\| \mathbf{D} \|_{\mathcal{W}}^2 = \| \mathbf{D} \|_{L^2(\Omega)^d}^2 + \| div\, \mathbf{D} \|_{L^2(\Omega)}^2, \| \phi \|_W = \| \nabla\phi \|_{L^2(\Omega)^d}$$

Since $meas(\Gamma_a) > 0$, the Friedrichs-Poincar? inequality is satisfied, thus,

$$\parallel \nabla\zeta \parallel_H \geq c_F \parallel \zeta \parallel_{H^1(\Omega)}, \quad \forall\zeta \in W, \tag{21}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a and $\nabla\zeta = (\zeta_{,i})$. It follows from (20) that $\|.\|_{H^1(\Omega)}$ and $\|.\|_W$ are equivalent norms on W and therefore $(W, \|.\|_W)$ is a real Hilbert space.

Moreover, by the Sobolev trace theorem, there exist a constant $\tilde{c}_0$ such that

$$\parallel \phi \parallel_{L^2(\Gamma_3)} \leq \tilde{c}_0 \parallel \phi \parallel_W, \quad \forall\phi \in W. \tag{22}$$

Moreover,recall that when $\boldsymbol{D} \in \mathcal{W}$ is a regular function, the following Green's type formula holds

$$(\boldsymbol{D}, \nabla\zeta)_H + (div\boldsymbol{D}, \zeta)_{L^2(\Omega)} = \int_\Gamma \boldsymbol{D} \cdot \boldsymbol{\nu}\zeta da, \quad \forall\zeta \in H^1(\Omega). \tag{23}$$

For any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \leq p \leq \infty$ and $k \geq 1$. For $T > 0$ we denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\parallel \boldsymbol{u} \parallel_{C(0,T;X)} = \max_{t\in[0,T]} \parallel \boldsymbol{u}(t) \parallel_X.$$

$$\parallel \boldsymbol{u} \parallel_{C^1(0,T;X)} = \max_{t\in[0,T]} \parallel \boldsymbol{u}(t) \parallel_X + \max_{t\in[0,T]} \parallel \dot{\boldsymbol{u}}(t) \parallel_X.$$

Consider two real Hilbert spaces X and H where the inclusion map from $(V, \|\cdot\|_X)$ to $(H, \|\cdot\|_H)$ is continuous and dense. Identifying the dual of H with itself, we can establish the Gelfand triplet $X \subset H \subset X'$. The notations $\|\cdot\|_X$, $\|\cdot\|_{X'}$, and $(\cdot,\cdot)_{X'\times X}$ represent the norms on X, X', and the duality pairing between X' and X, respectively.

Theorem 1 *Let $X \subset H \subset X'$ be a Gelfand triplet. Suppose $A: X \to X'$ is a hemicontinuous and monotone operator satisfying the following conditions:*

$$(Av, v)_{X'\times X} \geq w \parallel v \parallel_X^2 + \varsigma \quad \forall\, v \in X, \tag{24}$$

$$\parallel Av \parallel_{X'} \leq C(\parallel v \parallel_X + 1) \quad \forall\, v \in X, \tag{25}$$

where $w > 0$, $C > 0$, and $\varsigma \in \mathbb{R}$ are constants. Given an initial condition $u_0 \in H$ and a function $f \in L^2(0,T;X')$, there exists a unique function u satisfying:

$$u \in L^2(0,T;X) \cap C([0,T];H),$$

$$\dot{u} \in L^2(0,T;X'),$$

$$\dot{u}(t) + Au(t) = f(t) \quad \text{a.e.}\, t \in (0,T),$$

$$u(0) = u_0.$$

The previous abstract result can be found in [1, 2].

We assume in what follows that the viscosity operator $\mathcal{A}: \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies the following properties:

$$\begin{cases} (a)\text{There exists } L_{\mathcal{A}} > 0 \quad \text{such that} \\ \quad \| \mathcal{A}(\boldsymbol{x},\boldsymbol{\omega}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\omega}_2) \| \leq L_{\mathcal{A}} \| \boldsymbol{\omega}_1 - \boldsymbol{\omega}_2 \|, \\ \quad \text{for all } \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \text{a. e.} \boldsymbol{x} \in \Omega. \\ (b)\text{There exists } m_{\mathcal{A}} > 0 \quad \text{such that} \\ \quad (\mathcal{A}(\boldsymbol{x},\boldsymbol{\omega}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\omega}_2)) \cdot (\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \geq m_{\mathcal{A}} \| \boldsymbol{\omega}_1 - \boldsymbol{\omega}_2 \|^2, \\ \quad \text{for all } \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \text{a. e.} \boldsymbol{x} \in \Omega. \\ (c)\text{The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x},\boldsymbol{\omega}) \text{ is Lebesgue measurable on } \Omega, \\ \quad \text{for any } \boldsymbol{\omega} \in \mathbb{S}^d. \\ (d)\text{The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x},0) \text{ belongs to } \mathcal{H}. \end{cases} \tag{26}$$

The elasticity operator $\mathcal{B}: \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies the following properties:

$$\begin{cases} (a)\text{ There exists } L_{\mathcal{B}} > 0 \quad \text{suchthat} \\ \quad \| \mathcal{B}(\boldsymbol{x},\boldsymbol{\omega}_1) - \mathcal{B}(\boldsymbol{x},\boldsymbol{\omega}_2) \| \leq L_{\mathcal{B}} \| \boldsymbol{\omega}_1 - \boldsymbol{\omega}_2 \|, \\ \quad \text{for all } \boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \mathbb{S}^d, \text{a. e.} \boldsymbol{x} \in \Omega. \\ (b)\text{The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x},\boldsymbol{\omega}) \text{ is Lebesgue measurable on } \Omega, \\ \quad \text{for all } \boldsymbol{\omega} \in \mathbb{S}^d. \\ (c)\text{The mapping } \boldsymbol{x} \mapsto \mathcal{B}(\boldsymbol{x},0) \in \mathcal{H}. \end{cases} \tag{27}$$

The visco-plasticity operator $\mathcal{G}: \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ satisfies the following properties:

$$\begin{cases} (a)\text{ There exists a constant } L_{\mathcal{G}} > 0 \quad \text{such that} \\ \quad \|\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varsigma}_1,\alpha_1) - \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varsigma}_2,\alpha_2)\| \\ \qquad \leq L_{\mathcal{G}}(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varsigma}_1 - \boldsymbol{\varsigma}_2\| + \|\alpha_1 - \alpha_2\|), \\ \text{for all } t \in (0,T), \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varsigma}_1, \boldsymbol{\varsigma}_2 \in \mathbb{S}^d, \alpha_1, \alpha_2 \in \mathbb{R}^m, \text{a. e.} \boldsymbol{x} \in \Omega. \\ (b)\text{The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varsigma},\alpha) \text{ is Lebesgue measurable on } \Omega, \\ \quad \text{for all } \boldsymbol{\sigma},\boldsymbol{\varsigma} \in \mathbb{S}^d, \alpha \in \mathbb{R}^m, t \in (0,T). \\ (c)\text{The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x},\mathbf{0},\mathbf{0},\mathbf{0}) \in \mathcal{H}. \end{cases} \tag{28}$$

Electric permittivity operator $\boldsymbol{B} = (b_{ij}): \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} \text{(a) } \boldsymbol{B}(\boldsymbol{\varepsilon},E) = (b_{ij}(\boldsymbol{\varepsilon})E_j) \text{ for all } E = (E_i) \in \mathbb{R}^d, \text{a. e. } \boldsymbol{\varepsilon} \in \Omega. \\ \text{(b) } b_{ij} = b_{ji} \in L^\infty(\Omega), 1 \leq i,j \leq d. \\ \text{(c) There exists a constant } m_{\boldsymbol{B}} > 0 \text{ such that} \\ \qquad \boldsymbol{B}E.E \geq m_{\boldsymbol{B}} \| E \|^2, \text{for all } E = (E_i) \in \mathbb{R}^d, \text{a. e. in } \Omega. \end{cases} \tag{29}$$

The piezoelectric operator $\mathcal{E}: \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a) \quad \mathcal{E} = (f_{ijk}), f_{ijk} \in L^\infty(\Omega), 1 \leq i,j,k \leq d. \\ (b) \quad \mathcal{E}(\mathbf{x})\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{E}^*\boldsymbol{\tau}, \text{for all } \boldsymbol{\sigma} \in \mathbb{S}^d, \text{and all } \boldsymbol{\tau} \in \mathbb{R}^d. \end{cases} \tag{30}$$

The tangential function $p_e: \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$, $e = \nu, \tau$ satisfies

$$\begin{cases} \text{(a) There exists } L_e > 0 \text{ such that} \\ \quad \|p_e(\boldsymbol{x},\mu_1) - p_e(\boldsymbol{x},\mu_2)\| \leq L_e\|\mu_1 - \mu_2\| \\ \text{for all } \mu_1,\mu_2 \in \mathbb{R}, \text{a. e.} \boldsymbol{x} \in \Gamma_3 \\ \text{(d) For any } \mu \in \mathbb{R}, \boldsymbol{x} \mapsto p_e(\boldsymbol{x},\mu) \text{ is Lebesgue measurable on } \Gamma_3 \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p_e(\boldsymbol{x},0) \text{ belongs to } L^2(\Gamma_3). \end{cases} \tag{31}$$

The function $\Theta: \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_\Theta > 0 \quad \text{such that} \\ \|\Theta(x, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \alpha_1) - \Theta(x, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \alpha_2)\| \\ \quad \leq L_\Theta(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\alpha_1 - \alpha_2\|), \\ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}^m, \text{a.e.}\, x \in \Omega. \\ (b) \text{ The mapping } x \mapsto \Theta(x, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \alpha) \text{is Lebesgue measurable on } \Omega, \\ \forall \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \forall \alpha \in \mathbb{R}^m. \\ (c) \text{ The mapping } x \mapsto \Theta(x, 0,0,0) \in L^2(\Omega). \end{cases} \tag{32}$$

we assume that the mass density satisfies

$$\rho \in L^\infty(\Omega) \text{ there exists } \rho^* > 0 \text{ such that } \rho(x) > \rho^* \text{ and } x \in \Omega. \tag{33}$$

The volume force and surface traction are assumed to satisfy:

$$\boldsymbol{f}_0 \in C(0, T, L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C(0, T, L^2(\Gamma_2)^d). \tag{34}$$

$$q_0 \in C(0, T; L^2(\Omega)), \quad q_2 \in C\big(0, T; L^2(\Gamma_b)\big), \tag{35}$$

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{v}_0 \in H, \quad \boldsymbol{k}_0 \in Y. \tag{36}$$

We now proceed to the variational formulation of problem P. We employ an interior modified product on $H = L^2(\Omega)^d$ as follows

$$(u, v)_H = (\rho u, v)_H \quad \forall u, v \in H, \tag{37}$$

and the associated standard,

$$\| v \|_H = (\rho v, v)_H^{\frac{1}{2}} \quad \forall v \in H. \tag{38}$$

The hypotheses (37)-(38) imply that the norms $\|\cdot\|_H$ and $\|\cdot\|_H$ are equivalent on H. Moreover, the spaces $(V, \|\cdot\|_V)$ and $(V, \|\cdot\|_H)$ are included in each other densely and continuously. We denote the dual space of V by V'. By identifying H with its dual, we form the Gelfand triplet $V \subset H \subset V'$. The notation $(\cdot,\cdot)_{V' \times V}$ is used to represent the duality pairing between V' and V. We have:

$$(\boldsymbol{u}, \boldsymbol{v})_{V' \times V} = (\boldsymbol{u}, \boldsymbol{v})_H \quad \forall \boldsymbol{u} \in H, \ \forall \boldsymbol{v} \in V. \tag{39}$$

Then, we denote by $f: [0, T] \to V$ the function defined by

$$(\boldsymbol{f}(t), \boldsymbol{v})_{V' \times V} = \int_\Omega f_0(t) \cdot v \, dx + \int_{\Gamma_2} f_2(t) \cdot v \, da \quad \forall v \in V, \ t \in [0, T], \tag{40}$$

the function $q: [0, T] \to W$ defined by

$$(q(t), \zeta)_W = \int_\Omega q_0(t) \zeta dx - \int_{\Gamma_b} q_2(t) \zeta da. \tag{41}$$

Then, the functional $j: V \times V \to \mathbb{R}$ is defined by

$$j(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_3} p(u_\nu)v_\nu \, da \quad \forall \boldsymbol{u},\boldsymbol{v} \in V. \tag{42}$$

We observe that (40) entails

$$\boldsymbol{f} \in L^2(0,T;V'). \tag{43}$$

By using traditional methods based on Green's formula, we obtain the variational formulation for the problem (1)-(12).

Problem $\boldsymbol{PV}$

Find the displacement field $\boldsymbol{u}$: $[0;T] \to V$, the stress field $\boldsymbol{\sigma}$: $[0;T] \to H$,an electric potential field φ: $[0;T] \to W$, an electric displacement field $\boldsymbol{D}$: $[0;T] \to \mathcal{W}$, and an internal state variable field k: $[0;T] \to Y$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{F}\varepsilon(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\sigma(s) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s)), \varepsilon(\boldsymbol{u}(s)), k(s))ds, \tag{44}$$

$$(\ddot{\boldsymbol{u}}(t),\boldsymbol{v})_{V'\times V} + (\sigma(t),\varepsilon(v))_{\mathcal{Q}} + j(\dot{\boldsymbol{u}}(t),\boldsymbol{v}) = (\boldsymbol{f}(t),\boldsymbol{v})_{V'\times V} \quad \forall \boldsymbol{v} \in V, \tag{45}$$

$$\mathbf{D} = \mathcal{E}\varepsilon(\mathbf{u}) - \boldsymbol{B}\nabla(\varphi), \tag{46}$$

$$\left(\mathcal{E}\varepsilon(\boldsymbol{u}(t)) + \boldsymbol{B}\big(E(\varphi(t))\big), \nabla\phi\right)_H = (-q(t),\phi)_W, \forall \phi \in W, \tag{47}$$

$$\dot{\boldsymbol{k}}(t) = \phi(\sigma(t) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)), \varepsilon(\boldsymbol{u}(t)), \boldsymbol{k}(t)), \tag{48}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \boldsymbol{k}(0) = \boldsymbol{k}_0. \tag{49}$$

4. Existence and uniqueness

Theorem 2 *We assume that the conditions (26)-(43) are satisfied. Consequently, there is a unique solution* $(\boldsymbol{u},\boldsymbol{\sigma},\varphi,\boldsymbol{k},\boldsymbol{D})$ *to problem* PV*. Moreover, this solution satisfies*

$$\boldsymbol{u} \in W^{1,2}(0,T;V) \cap C^1(0,T;H) \quad \ddot{u} \in L^2(0,T;V'), \tag{50}$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \quad Div\boldsymbol{\sigma} \in L^2(0,T;V'), \tag{51}$$

$$\varphi \in C(0,T;W), \tag{52}$$

$$\boldsymbol{k} \in W^{1,2}(0,T;Y), \tag{53}$$

$$\boldsymbol{D} \in C(0,T;\mathcal{W}). \tag{54}$$

In the first step, we consider the following auxiliary problem, where the function $\boldsymbol{\eta} = (\eta^1,\eta^2) \in L^2(0,T;V' \times Y)$ is given.

Problem $\boldsymbol{\mathcal{P}_\eta^1}$

Find a displacement field $\boldsymbol{u}_\eta$: $(0,T) \to V$, such that

$$\begin{aligned}\left(\ddot{\boldsymbol{u}}_\eta(t),v\right)_{V'\times V} + \left(\mathcal{A}\varepsilon\big(\dot{\boldsymbol{u}}_\eta(t)\big),\varepsilon(v)\right)_{\mathcal{H}} + (\eta^1(t),\boldsymbol{v})_{V'\times V} + j(\dot{\boldsymbol{u}}(t),\boldsymbol{v}) \\ = (f(t),\boldsymbol{v})_{V'\times V} \quad \forall \boldsymbol{v} \in V, \text{a.e.}\, t \in [0,T].\end{aligned} \tag{55}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}_\eta(0) = \boldsymbol{v}_0. \tag{56}$$

Regarding the $\mathcal{P}_\eta^1$ problem, we have the following result.

Lemma 3 *The* $\mathcal{P}_\eta^1$ *problem admits a unique solution that satisfies the regularity condition (50). However, if* u_i *represents the solution of problem* $\mathcal{P}_\eta^1$, *for* $\eta^1 = \eta_i^1 \in L^2(0,T;V')$, $i = 1,2$, *then there exists a constant* $C > 0$ *such that*

$$\int_0^t \| \dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s) \|_V^2 \ ds \leq C \int_0^t \| \boldsymbol{\eta}_1^1(s) - \boldsymbol{\eta}_2^1(s) \|_{V'}^2 \ ds \quad \forall t \in [0,T]. \tag{57}$$

Proof. We define the operator $A: V \to V'$ in the following manner:

$$(A\boldsymbol{u}, \boldsymbol{u})_{V'\times V} = (\mathcal{A}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + j(\boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{58}$$

By letting $\dot{\boldsymbol{u}}_\eta = \boldsymbol{v}_\eta$, the $\mathcal{P}_\eta^1$ problem can be reformulated as follows: Find the displacement field $v_\eta: [0,T] \to V$ such that:

$$\dot{\boldsymbol{u}}_\eta(t) + A\boldsymbol{v}_\eta(t) + \eta(t) = f(t), \tag{59}$$

with the initial condition:

$$\boldsymbol{v}_\eta(0) = \boldsymbol{v}_0. \tag{60}$$

From equations (14)-(16), (26) (a), (31) (a), (42), and (58), we can deduce that

$$|A\boldsymbol{u} - A\boldsymbol{v}|_{V'} \leq (L_{\mathcal{A}} + L_p C_0^2) \| \boldsymbol{u} - \boldsymbol{v} \|_V \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{61}$$

This shows that the operator $A: V \to V'$ is Lipschitz continuous, which guarantees its continuity. Consequently, the mapping $t \mapsto A(u + tv)$ is continuous, this implies that A is a hemicontinuous operator.

Next, utilizing equations (58), (26) (c), and (31) (b), we establish the following inequality:

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_{V'\times V} \geq m_A \| \boldsymbol{u} - \boldsymbol{v} \|_V^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{62}$$

This demonstrates that A is a monotonic operator.

By substituting $v = 0_V$ into (62) and applying the inequality $\alpha\beta \leq \frac{\alpha^2}{2} + \frac{\beta^2}{2}$, we obtain:

$$(A\boldsymbol{u}, \boldsymbol{u})_{V'\times V} \geq m_{\mathcal{A}} \| \boldsymbol{u} \|_V^2 - \|A\boldsymbol{0}_V\|_{V'} \| \boldsymbol{u} \|_V$$

$$\geq \frac{1}{2} m_{\mathcal{A}} \| \boldsymbol{u} \|_V^2 - \frac{1}{2m_{\mathcal{A}}} \|A0_V\|_{V'}^2, \quad \forall \boldsymbol{u} \in V.$$

Thus,

$$(A\boldsymbol{u}, \boldsymbol{u})_{V'\times V} \geq \lambda \| \boldsymbol{u} \|_V^2 + \alpha \quad \forall \boldsymbol{u} \in V,$$

where $\lambda = \frac{1}{2} m_{\mathcal{A}}$ and $\alpha = -\frac{1}{2m_{\mathcal{A}}} |A0_V|_{V'}^2$.

This confirms that condition (1) of Theorem 1 is satisfied. Furthermore, by setting $\boldsymbol{v} = \boldsymbol{0}_V$ in (61), we find:

$$\| A\boldsymbol{u} \|_{V'} \leq c(\| \boldsymbol{u} \|_V + 1) \quad \forall \boldsymbol{u} \in V.$$

Furthermore, recalling from equations (36) and (37) that $f - \eta \in L^2(0,T;V')$ and $\boldsymbol{v}_0 \in H$, Theorem 1 guarantees the existence of a unique function v that fulfills the following conditions:

$$\boldsymbol{v}_\eta \in L^2(0,T;V') \cap C(0,T;H), \quad \dot{\boldsymbol{v}}_\eta \in L^2(0,T;V'), \tag{63}$$

$$\dot{\boldsymbol{v}}_\eta(t) + A\boldsymbol{v}_\eta(t) + \eta(t) = f(t) \quad \text{p. p.}\, t \in [0,T], \tag{64}$$

$$\boldsymbol{v}_\eta(0) = \boldsymbol{v}_0. \tag{65}$$

Next, define $\boldsymbol{u}: [0,T] \to V$ as follows:

$$u_\eta(t) = \int_0^t \boldsymbol{v}_\eta(s)\, ds + \boldsymbol{u}_0 \quad \forall t \in [0,T]. \tag{66}$$

From equations (58) and (63)-(66), it can be concluded that $\boldsymbol{u}$ is a solution to the variational problem $\mathcal{P}_\eta^1$ and meets the regularity condition (50). This completes the proof of the existence part of Lemma 3.

The uniqueness of the solution is derived from the uniqueness results established for problems (63)-(66), as guaranteed by Theorem 1. Consider $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ in $L^2(0,T;V')$, and let $\boldsymbol{u}_i = \boldsymbol{u}_{\eta_i}$ and $\boldsymbol{v}_i = \dot{\boldsymbol{u}}_{\eta_i}$ for $i = 1,2$. We obtain the following from equation (55):

$$\begin{aligned}(\dot{\boldsymbol{v}}_1 - \dot{\boldsymbol{v}}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2)_{V'\times V} + (\mathcal{A}\varepsilon(\boldsymbol{v}_1) - \mathcal{A}\varepsilon(\boldsymbol{v}_2), \varepsilon(\boldsymbol{v}_1) - \varepsilon(\boldsymbol{v}_2))_{\mathcal{H}} \\ = -(\eta_1 - \eta_2, \boldsymbol{v}_1 - \boldsymbol{v}_2)_{V'\times V}.\end{aligned} \tag{67}$$

By integrating the above equality with respect to t and using the initial conditions $\boldsymbol{v}_1(0) = \boldsymbol{v}_2(0) = \boldsymbol{v}_0$, along with the properties of the operator $\mathcal{A}$, we obtain

$$m_{\mathcal{A}} \int_0^t \| \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s) \|_V^2 \, ds \le -\int_0^t (\eta_1(s) - \eta_2(s), \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s))_{V'\times V}\, ds. \tag{68}$$

Now, employing inequality $\alpha\beta \le \frac{\alpha^2}{2} + \frac{\beta^2}{2}$ and its consequences, we infer that

$$\int_0^t \| \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s) \|_V^2 \, ds \le C \int_0^t \| \boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s) \|_{V'}^2 \, ds. \tag{69}$$

Problem $\boldsymbol{\mathcal{P}_\eta^2}$

Find an electrical potential $\varphi_\eta: (0,T) \to W$ such that

$$\left(\mathcal{E}\varepsilon\big(\boldsymbol{u}_\eta(t)\big) + \mathbf{B}\left(E\big(\varphi(t)_\eta\big)\right), \nabla\phi\right)_H = (-q(t), \phi)_W, \forall \phi \in W. \tag{70}$$

We have the following result for problem $\mathcal{P}_\eta^2$

Lemma 4 *Problem (70) has unique solution φ_η which satisfies the regularity (52). Moreover, if φ_η represents the solution to Problem $\mathcal{P}_\eta^2$ for η_i, $i = 1,2$, then there exists $C > 0$ such that*

$$\|\varphi_1(t) - \varphi_2(t)\|_W \le C\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V, \quad \forall t \in (0,T). \tag{71}$$

Proof. Consider the form $G: W \times W \to \mathbb{R}$ defined by

$$G(\varphi, \phi) = (\mathbf{B}\nabla\varphi, \nabla\phi)_H \quad \forall \varphi, \phi \in W. \tag{72}$$

Using equations (20), (21), (29), and (72), we demonstrate that the form G is bilinear, continuous, symmetric, and coercive on W. Additionally, by employing equation (41) and the Riesz representation theorem, we can define an element $\xi_\eta: [0,T] \to W$ such that

$$\big(\xi_\eta(t), \phi\big)_W = (q(t), \phi)_W + \big(\mathcal{E}\varepsilon\big(\boldsymbol{u}_\eta(t)\big), \nabla\phi\big)_H \quad \forall \phi \in W, t \in (0,T).$$

Applying the Lax-Milgram Theorem, we deduce that there exists a unique element $\varphi_\eta(t) \in W$ such that

$$G(\varphi_\eta(t), \phi) = (\xi_\eta(t), \phi)_W \quad \forall \phi \in W. \tag{73}$$

From equation (72), it follows that φ_η is a solution to equation (70). Let $\varphi_{\eta_i} = \varphi_i$ and $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$ for $i = 1,2$. Using equation (70), we obtain

$$\|\varphi_1(t) - \varphi_2(t)\|_W \le C\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \quad \forall t \in (0, T).$$

Since $\boldsymbol{u}_\eta \in C^1(0, T; V)$, it implies that $\varphi_\eta \in C(0, T; W)$. This completes the proof.

Now, define $\boldsymbol{k}_\eta \in L^2(0, T; Y)$ by

$$\boldsymbol{k}_\eta(t) = \boldsymbol{k}_0 + \int_0^t \boldsymbol{\eta}^2(s)ds. \tag{74}$$

In the fourth step we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 3 and $\boldsymbol{k}_\eta$ defined in (74) to consider the following Cauchy problem for the stress field.

Problem $\boldsymbol{\mathcal{P}_{\eta,\sigma}}$

Find the stress field $\boldsymbol{\sigma}_\eta: (0, T) \to \mathbb{S}_n$ which is a solution of the problem

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{B}\left(\varepsilon(\boldsymbol{u}_\eta(t))\right) + \int_0^t \mathcal{G}\left(\boldsymbol{\sigma}_\eta(s), \varepsilon(\boldsymbol{u}_\eta(s), \boldsymbol{k}_\eta(s))\right) ds, \quad \forall t \in [0, T]. \tag{75}$$

Lemma 5 *There exists a unique solution of Problem $\mathcal{P}_{\eta,\sigma}$ and it satisfies (51). Moreover, if $\boldsymbol{u}_{\eta_i}, \theta_{\eta_i}$ and σ_{η_i} represent the solutions of problems $\mathcal{P}_\eta^1, \mathcal{P}_\eta^2$ and $\mathcal{P}_\eta$, respectively, for $i = 1,2$, then there exists $C >$ 0 such that*

$$\begin{aligned}\left\|\boldsymbol{\sigma}_{\eta_1}(t) - \boldsymbol{\sigma}_{\eta_2}(t)\right\|_{\mathcal{H}}^2 \le C \Big(&\left\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\right\|_V^2 \\ &+ \int_0^t \left\|\boldsymbol{u}_{\eta_1}(s) - \boldsymbol{u}_{\eta_2}(s)\right\|_V^2 + \left\|\boldsymbol{k}_{\eta_1}(t) - \boldsymbol{k}_{\eta_2}(t)\right\|_Y^2 ds\Big).\end{aligned} \tag{76}$$

Proof. Let $\mathcal{T}_\eta: L^2(0, T; \mathcal{H}) \to L^2(0, T; \mathcal{H})$ be the operator given by

$$\mathcal{T}_\eta\boldsymbol{\sigma}(t) = \mathcal{B}\left(\varepsilon(\boldsymbol{u}_\eta(t))\right) + \int_0^t \mathcal{G}\left(\boldsymbol{\sigma}(s), \varepsilon(\boldsymbol{u}_\eta(s), \boldsymbol{k}_\eta(s))\right) ds, \quad \forall t \in [0, T]. \tag{77}$$

For $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in L^2(0, T; \mathcal{H})$, we use (77) and (28) to obtain for all $t \in [0, T]$

$$\left\|\mathcal{T}_\eta\boldsymbol{\sigma}_1(t_1) - \mathcal{T}_\eta\boldsymbol{\sigma}_2(t_1)\right\|_{\mathcal{H}}^2 \le L_{\mathcal{G}}^2 T \int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 ds.$$

Integration on the time interval $(0, t_2) \subset (0, T)$, it follows that

$$\int_0^{t_2} \left\|\mathcal{T}_\eta\boldsymbol{\sigma}_1(t_1) - \mathcal{T}_\eta\boldsymbol{\sigma}_2(t_1)\right\|_{\mathcal{H}}^2 dt_1 \le L_{\mathcal{G}}^2 T \int_0^{t_2} \int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 \, ds \, dt_1$$

Therefore,

$$\left\|\mathcal{T}_\eta\boldsymbol{\sigma}_1(t_2) - \mathcal{T}_\eta\boldsymbol{\sigma}_2(t_2)\right\|_{\mathcal{H}}^2 \le L_{\mathcal{G}}^4 T^2 \int_0^{t_2} \int_0^{t_1} \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 \, ds \, dt_1.$$

For $t_1, t_2, \ldots, t_p \in (0, T)$, we generalize the procedure above by recurrence on p. We obtain the inequality

$$\left\|\mathcal{T}_\eta\boldsymbol{\sigma}_1(t_p)-\mathcal{T}_\eta\boldsymbol{\sigma}_2(t_p)\right\|_{\mathcal{H}}^2 \\ \leq L_{\mathcal{G}}^{2p}T^p\int_0^{tp}\cdots\int_0^{t_2}\int_0^{t_1}\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2 ds dt_1 \dots dt_{p-1}.$$

Which implies

$$\left\|\mathcal{T}_\eta\boldsymbol{\sigma}_1(t_p)-\mathcal{T}_\eta\boldsymbol{\sigma}_2(t_p)\right\|_{\mathcal{H}}^2 \leq \frac{L_{\mathcal{G}}^{2p}T^{p+1}}{p!}\int_0^T\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}^2\,ds.$$

Thus, we can infer, by integrating over the interval time $(0,T)$, that

$$\left\|\mathcal{T}_\eta\boldsymbol{\sigma}_1-\mathcal{T}_\eta\boldsymbol{\sigma}_2\right\|_{L^2(0,T;\mathcal{H})}^2 \leq \frac{L_{\mathcal{G}}^{2p}T^{p+2}}{p!}\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|_{L^2(0,T;\mathcal{H})}^2.$$

It follows from this inequality that for sufficiently large p, the operator $\mathcal{T}_\eta^{(p)}$ is a contraction on the Banach space $L^2(0,T;\mathcal{H})$. Consequently, there exists a unique element $\boldsymbol{\sigma}_\eta\in L^2(0,T;\mathcal{H})$ such that $\mathcal{T}_\eta^{(p)}\boldsymbol{\sigma}_\eta=\boldsymbol{\sigma}_\eta$. Furthermore, $\boldsymbol{\sigma}_\eta$ is the unique solution to Problem $\mathcal{P}_\eta$. Given the regularity of $\boldsymbol{u}_\eta$ and the properties of the operators $\mathcal{B}$ and $\mathcal{G}$, it follows that $\boldsymbol{\sigma}_\eta\in L^2(0,T;\mathcal{H})$.

Now, consider $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\in L^2(0,T;V'\times Y)$, and for $i=1,2$, denote $\boldsymbol{u}_{\eta_i}=\boldsymbol{u}_i$, $\boldsymbol{\sigma}_{\eta_i}=\boldsymbol{\sigma}_i$, and $\boldsymbol{k}_{\eta_i}=\boldsymbol{k}_i$. We have

$$\boldsymbol{\sigma}_i(t)=\mathcal{B}\varepsilon(\boldsymbol{u}_i(t))+\int_0^t\mathcal{G}(\boldsymbol{\sigma}_i(s),\varepsilon(\boldsymbol{u}_i(s)),\boldsymbol{k}_i)\,ds,\quad \text{a.e. } t\in(0,T).$$

Using the properties (27) and (28) of $\mathcal{B}$ and $\mathcal{G}$, we find

$$\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal{H}}^2 \\ \leq C\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2+\int_0^t\|\sigma_1(s)-\sigma_2(s)\|_{\mathcal{H}}^2\,ds \\ +\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\,ds+\int_0^t\|\boldsymbol{k}_1(s)-\boldsymbol{k}_2(s)\|_Y^2\,ds\Big),\quad\forall\, t\in[0,T]. \tag{78}$$

We use Gronwall argument in the previous inequality to deduce (76), which concludes the proof of Lemma 5.

Finally, as a consequence of these results and using the properties of the operator $\mathcal{G}$ the operator $\mathcal{E}$, the function S for $t\in(0,T)$, we consider the element

$$\Lambda\boldsymbol{\eta}(t)=(\Lambda^1\boldsymbol{\eta}(t),\Lambda^2\boldsymbol{\eta}(t))\in V'\times Y, \tag{79}$$

defined by

$$(\Lambda^1\boldsymbol{\eta}(t),\boldsymbol{v})_{V'\times Y}=\left(\mathcal{E}^*\nabla\varphi_\eta(t),\boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathcal{H}} \\ +\left(\int_0^t\mathcal{G}\big(\boldsymbol{\sigma}_\eta(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)),\boldsymbol{k}_{\boldsymbol{\eta}}(s)\big)ds,\boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathcal{H}},\forall\boldsymbol{v}\in V, \tag{80}$$

$$\Lambda^2\boldsymbol{\eta}(t)=\Phi\big(\boldsymbol{\sigma}_\eta,\varepsilon(\boldsymbol{u}_\eta(t)),\boldsymbol{k}_{\boldsymbol{\eta}}(t)\big). \tag{81}$$

In this context, for any $\boldsymbol{\eta}\in L^2(0,T;V'\times Y)$, the functions $\boldsymbol{u}_\eta$, φ_η, and $\boldsymbol{k}_\eta$ denote the displacement field, the electric potential field, and the stress field, respectively, as derived in Lemmas 3, 4, and 5. Additionally, $\boldsymbol{k}_\eta$ represents the internal state variable given by (74). The following result can be stated.

Lemma 6 *The mapping Λ has a fixed point $\boldsymbol{\eta}^*\in L^2(0,T;V'\times Y)$, such that $\Lambda\boldsymbol{\eta}^*=\boldsymbol{\eta}^*$.*

Proof. Let $t \in (0,T)$ and $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in L^2(0,T;V' \times Y)$. We use the notation that $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\eta_i} = \dot{\boldsymbol{u}}_i$, $\varphi_{\eta_i} = \varphi_i$, $\boldsymbol{k}_{\eta_i} = \boldsymbol{k}_i$ and $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$ for $i = 1,2$. Using (14)-(15),(28), (30) and (32) to find

$$\begin{aligned} &\| \Lambda(\boldsymbol{\eta}_1)(t) - \Lambda(\boldsymbol{\eta}_2)(t) \|^2_{V' \times Y} \\ \leq\ & C(\|\varphi_1(t) - \varphi_2(t)\|^2_W + \int_0^t (\|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}} + \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|^2_V \\ &+ \|\boldsymbol{k}_1(s) - \boldsymbol{k}_2(s)\|^2_Y ds) \\ &+ \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|^2_{\mathcal{H}} + \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|^2_V + \|\boldsymbol{k}_1(s) - \boldsymbol{k}_2(s)\|^2_Y), \end{aligned} \tag{82}$$

we use estimates (76), (71) to obtain

$$\begin{aligned} &\| \Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t) \|^2_{V' \times Y} \\ \leq\ & C(\|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|^2_V + \|\boldsymbol{k}_1(s) - \boldsymbol{k}_2(s)\|^2_Y \\ &+ \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|^2_V + \|\boldsymbol{k}_1(s) - \boldsymbol{k}_2(s)\|^2_Y ds. \end{aligned} \tag{83}$$

Since $\boldsymbol{u}_i(t) = \int_0^t \dot{\boldsymbol{u}}_i(s) ds + \boldsymbol{u}_0, \forall t \in (0,T)$, we have

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_V \leq \int_0^T \|\dot{\boldsymbol{u}}_1(s) - \dot{\boldsymbol{u}}_2(s)\|^2_V ds. \tag{84}$$

Combining (84) and (57), and using the Gronwall's inequality, we have

$$\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \leq C \int_0^t \left\|\boldsymbol{\eta}_1^1 - \boldsymbol{\eta}_2^1\right\|_{V'} ds, \quad t \in (0,T). \tag{85}$$

Furthermore, from (74) we have

$$\|\boldsymbol{k}_1(t) - \boldsymbol{k}_2(t)\|^2_Y \leq C \int_0^t \left\|\boldsymbol{\eta}_1^2(s) - \boldsymbol{\eta}_2^2(s)\right\|^2_Y ds. \tag{86}$$

Form the previous inequality and estimates (85) and (83) it follows now that

$$\begin{aligned} &\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|^2_{V' \times Y} \\ &\leq C \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|^2_{V' \times Y} ds. \end{aligned} \tag{87}$$

Let is introduce the following notations

$$\begin{cases} I_1 = \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{V' \times Y} ds, \\ \vdots \\ I_k = \int_0^t \int_0^{s_{k-1}} \cdots \int_0^{s_1} \|\boldsymbol{\eta}_1(r) - \boldsymbol{\eta}_2(r)\|_{V' \times Y}, \end{cases}$$

and by induction, by denoting by Λ^m the m power of the operator Λ, we obtain

$$\begin{aligned} &\|\Lambda^m\boldsymbol{\eta}_1(t) - \Lambda^m\boldsymbol{\eta}_2(t)\|_{V' \times Y} \\ &\qquad \leq C^m \left(\sum_{k=1}^m C_m^k I^{m-k} \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_{V' \times Y}\right), \end{aligned}$$

for all $t \in (0,T)$ and $m \in \mathbb{N}$,

$$\begin{aligned} I^{m-k} \| \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2 \|_{V' \times Y} &= \int_{(m-k)\text{fois}} \cdot \int \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{V' \times Y} \\ &\leq \int_0^s \int \cdots \int_{(m-k)\text{fois}} \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{L^2(0,T;V' \times Y)} \\ &\leq \frac{t^{m-k}}{k!} \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{L^2(0,T;V' \times Y)} \\ &\leq \frac{T^{m-k}}{k!} \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{L^2(0,T;V' \times Y)}, \end{aligned}$$

$$\begin{aligned}&\| \Lambda^m \boldsymbol{\eta}_1(t) - \Lambda^m \boldsymbol{\eta}_2(t) \|_{L^2(0,T;V'\times Y)}\\ &\qquad \le C^m \left(\sum_{k=1}^{m} C_m^k \frac{T^{m-k}}{k!} \|\eta_1(t) - \eta_2(t)\|_{L^2(0,T;V'\times Y)} \right)\\ &\qquad \le \frac{(CT)^m}{m!} \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|^2_{L^2(0,T;V'\times Y)},\end{aligned}$$

This implies that for sufficiently large m, the operator Λ^m is a contraction on the Banach space $L^2(0,T;V' \times Y)$. Consequently, Λ^m possesses a unique fixed point $\boldsymbol{\eta}^* \in L^2(0,T;V' \times Y)$, which means that $\boldsymbol{\eta}^*$ is also a unique fixed point of Λ.

Existence

Let $\boldsymbol{\eta}^* \in L^2(0,T;V' \times Y)$ be the fixed point of Λ. We define

$$\boldsymbol{u} = \boldsymbol{u}_{\eta^*}, \quad \boldsymbol{k} = \boldsymbol{k}_{\eta^*}, \quad \varphi = \varphi_{\eta^*}, \tag{88}$$

$$\boldsymbol{\sigma} = \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{E}^*\nabla\varphi(t) + \boldsymbol{\sigma}_{\eta^*}, \tag{89}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{B}\nabla\varphi. \tag{90}$$

We demonstrate that the tuple $(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{k}, \varphi, \boldsymbol{D})$ satisfies the conditions outlined in (44)-(49) and (50)-(54). Specifically, by substituting $\boldsymbol{\eta}^* = \boldsymbol{\eta}$ into (75) and applying (88)-(89), we confirm that (44) is satisfied. Next, we consider (55) for $\boldsymbol{\eta}^* = \boldsymbol{\eta}$ and use (88) to derive

$$\begin{aligned}&(\ddot{\boldsymbol{u}}(t), \boldsymbol{v})_{V'\times V} + (\mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(t)), \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (\eta^{1*}(t), \boldsymbol{v})_{V'\times V} + j(\dot{\boldsymbol{u}}(t), \boldsymbol{v})\\ &\quad = (\boldsymbol{f}(t), \boldsymbol{v})_{V'\times V} \quad \forall \boldsymbol{v} \in V, \text{a.e.}\, t \in [0,T].\end{aligned} \tag{91}$$

The relationships $\Lambda^1(\boldsymbol{\eta}^*) = \boldsymbol{\eta}^{1*}$ and $\Lambda^2(\boldsymbol{\eta}^*) = \boldsymbol{\eta}^{2*}$, together with (80)-(81), (88), and (89), imply that for all $\boldsymbol{v} \in V$,

$$\begin{aligned}&(\boldsymbol{\eta}^{\boldsymbol{1}^*}(t), \boldsymbol{v})_{V'\times Y} = (\mathcal{B}(\varepsilon\boldsymbol{u}(t)), \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi(t), \varepsilon(\boldsymbol{v}))_{\mathcal{H}},\\ &\quad + \left(\int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s)) - \mathcal{E}^*\nabla\varphi(t), \varepsilon(\boldsymbol{u}(s)), \theta(s), \boldsymbol{k}(s))\, ds, \varepsilon(\boldsymbol{v}) \right)_{\mathcal{H}},\end{aligned} \tag{92}$$

$$\boldsymbol{\eta}^{\boldsymbol{2}^*}(t) = \Phi(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\boldsymbol{u}}(s)) - \mathcal{E}^*\nabla\varphi(t), \varepsilon(\boldsymbol{u}(t)), \theta(t), \boldsymbol{k}). \tag{93}$$

From (93) and (74), it follows that (48) is satisfied. By substituting (92) into (91) and using (44), we verify that (45) holds.

We then substitute $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ into (70) and use (88) to derive (47). Furthermore, (49), along with the regularities given by (50), (52), and (53), follow from Lemmas 3, 4, and the relationship in (74). The regularity $\boldsymbol{\sigma} \in C(0,T;\mathcal{H})$ is ensured by Lemmas 5.

Considering $t_1, t_2 \in [0,T]$, from (21), (29), (30), and (90), we conclude that there exists a positive constant $C > 0$ such that

$$\|\boldsymbol{D}(t_1) - \boldsymbol{D}(t_2)\|_H \le C(\|\varphi(t_1) - \varphi(t_2)\|_W + \|\boldsymbol{u}(t_1) - \boldsymbol{u}(t_2)\|_V).$$

The regularity of $\boldsymbol{u}$ and φ given by (50) and (52) implies that

$$\boldsymbol{D} \in C(0,T;H). \tag{94}$$

By choosing $\phi \in D(\Omega)^d$ in (46)-(47) and using (41), we find

$$div\boldsymbol{D}_*(t) = q_0(t), \quad \forall t \in [0,T]. \tag{95}$$

The property (54) follows from (35), (94), and (95), thereby concluding the existence part of the Theorem.

Uniqueness

The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ.

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